Computation of DFT using FFT Algorithms

FFT : Fast Fourier Transform Algorithms

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Discrete Fourier Transform

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad k = 0,1,2.....N-1$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} \qquad n = 0,1,2.....N-1$$

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For SPECTRAL ANALYSIS and LINEAR FILTERING
Computation of DFT involves large no. of computations

Efficient algorithms are developed which require less no. of computations → Fast Fourier Transform Algorithms

or

FFT Algorithms

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Direct Computation of DFT :

To compute one DFT point :

N complex multiplications are required

N-1 complex additions are required

To compute N DFT points (k = 0,1, …,N-1)

N^2 complex multiplications (N x N)

N^2-N complex additions. ((N-1) x N)

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Each complex multiplication requires :

$(a + jb) \times (c + jd)$
$(ac - bd) + j(ad + bc)$

4 real multiplications & 2 real additions

N^2 complex multipications require :

$4N^2$ real multiplications
$2N^2$ real additions

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Each complex addition requires :

$(a + jb) + (c + jd)$
$(a+c) + j(b+d)$

2 real additions

N^2 - N complex additions require :

$2(N^2 - N) = \underline{2N^2 - 2N}$ real additions

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$4N^2$ real multiplications
$2N^2$ real additions

$2N^2 - 2N$ real additions

Total

Real Multiplications	: $4N^2$
Real Additions	: $4N^2 - 2N$
Trigonometric functions	: $2N^2$

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DFT $$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{kn} \qquad k=0,1,2.....N-1$$

where $$W_N=e^{-j2\pi/N}$$

W_N is called as Phase factor or Twiddle factor

IDFT $$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-kn} \qquad n=0,1,2.....N-1$$

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Properties of W_N

Periodicity property :

$$W_N^{k+N} = W_N^k \qquad W_8^{k+8} = W_8^k$$

W_N^k is periodic with period N

Symmetry property :

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

Relation :

$$W_N^2 = W_{N/2}$$

$$W_N^2 = e^{-2j2\pi/N} = e^{-j2\pi/N/2} = W_{N/2}$$

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Radix - 2 Decimation in Time FFT algorithm : (Radix – 2 DIT – FFT algorithm)

N is considered as integer power of 2

N-point data sequence, x(n), is splitted into
two N/2 point data sequences
containing even numbered and odd numbered data points

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

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$$X(k) = \sum_{n-even} x(n) W_N^{nk} + \sum_{n-odd} x(n) W_N^{nk}$$

Substitute n=2r for even {x(0), x(2), x(4), x(6)}

n=2r+1 for odd {x(1), x(3), x(5), x(7)}

$$X(k) = \sum_{r=0}^{N/2-1} x(2r) W_N^{2rk} + \sum_{r=0}^{N/2-1} x(2r+1) W_N^{(2r+1)k}$$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r) (W_N^2)^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1) (W_N^2)^{rk}$$

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$$W_N^2 = W_{N/2}$$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{N/2-1} x(2r+1) W_{N/2}^{rk}$$

$$X(k) = G(k) + W_N^k H(k)$$

where G(k) & H(k) are N/2 point DFTs of x(2r) & x(2r+1)

Since G(k) & H(k) are periodic with period N/2

we have, G(k+N/2) = G(k) & H(k+N/2) = H(k)

also, $W_N^{k+N/2} = -W_N^k$

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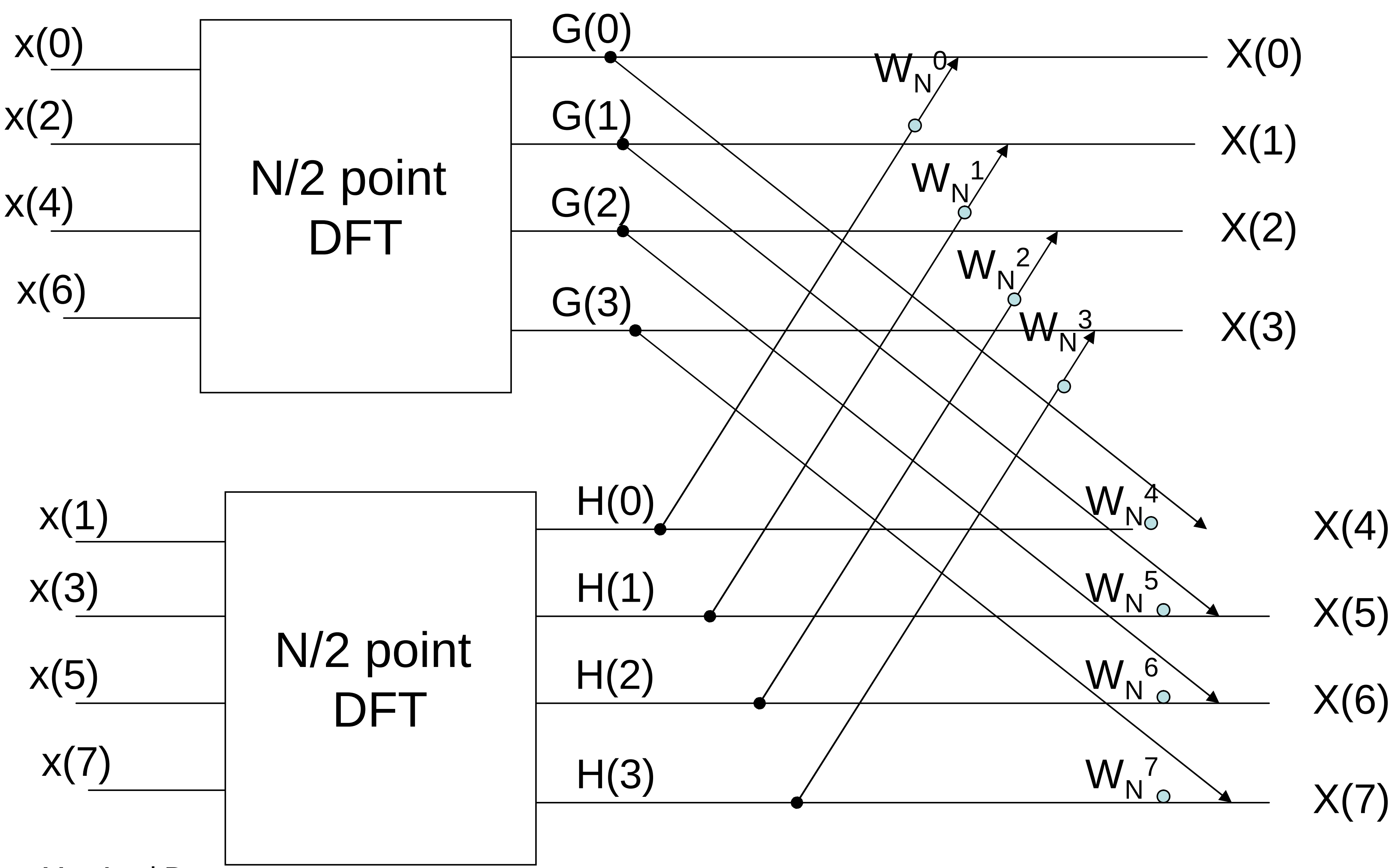
$$X(k) = G(k) + W_N^k H(k)$$ k = 0,1,…..,N/2 - 1

$$X(k + \frac{N}{2}) = G(k) - W_N^k H(k)$$ k = 0,1,…..,N/2 - 1

Signal Flow Graph



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No. of Complex Multiplications & Additions:

2 N/2 pt. DFTs : $2(N/2)^2$

Combining DFTs : N ($H(k) \times W_N^k$)

Total $N + 2(N/2)^2 \sim N^2/2$ (Complex Mul. & Add.)

Reduction by a factor of 2

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G(k) & H(k) can be computed by splitting each of them into 2 N/4 pt. DFTs

$$G(k) = \sum_{r=0}^{N/2-1} g(r) W_{N/2}^{rk}$$

$$G(k) = \sum_{l=0}^{N/4-1} g(2l) W_{N/2}^{2lk} + \sum_{l=0}^{N/4-1} g(2l+1) W_{N/2}^{(2l+1)k}$$

$$G(k) = \sum_{l=0}^{N/4-1} g(2l) (W_{N/2}^{2})^{lk} + W_{N/2}^{k} \sum_{l=0}^{N/4-1} g(2l+1) (W_{N/2}^{2})^{lk}$$

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$$W_{N/2}^{2} = W_{N/4}$$

$$G(k) = \sum_{l=0}^{N/4-1} g(2l) W_{N/4}^{lk} + W_{N/2}^{k} \sum_{l=0}^{N/4-1} g(2l+1) W_{N/4}^{lk}$$

$$H(k) = \sum_{l=0}^{N/4-1} h(2l) W_{N/4}^{lk} + W_{N/2}^{k} \sum_{l=0}^{N/4-1} h(2l+1) W_{N/4}^{lk}$$

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$$W_{N/2} = W_N^2$$

$$G(k) = \sum_{l=0}^{N/4-1} g(2l) W_{N/4}^{lk} + W_N^{2k} \sum_{l=0}^{N/4-1} g(2l+1) W_{N/4}^{lk}$$

$$H(k) = \sum_{l=0}^{N/4-1} h(2l) W_{N/4}^{lk} + W_N^{2k} \sum_{l=0}^{N/4-1} h(2l+1) W_{N/4}^{lk}$$

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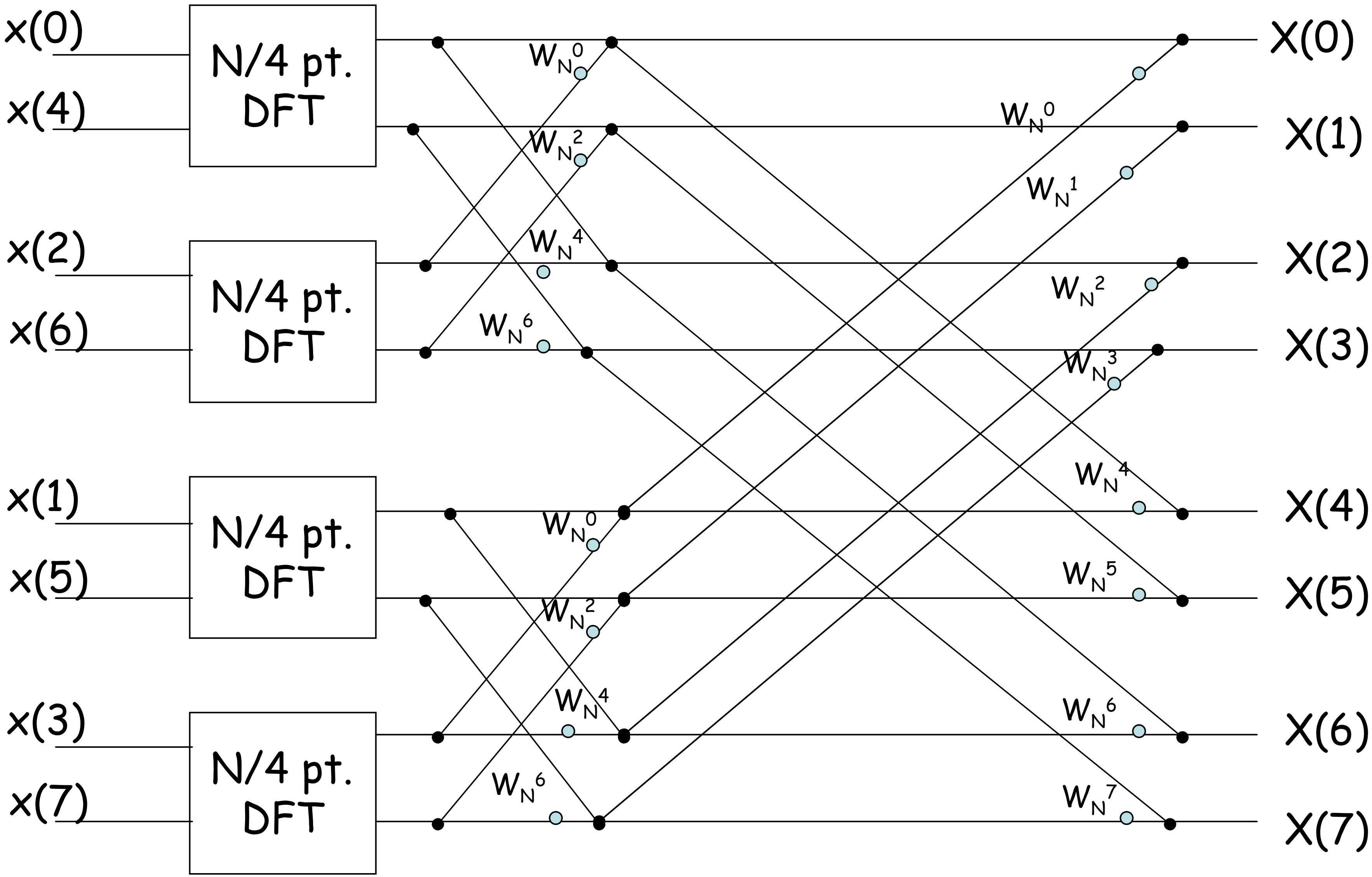

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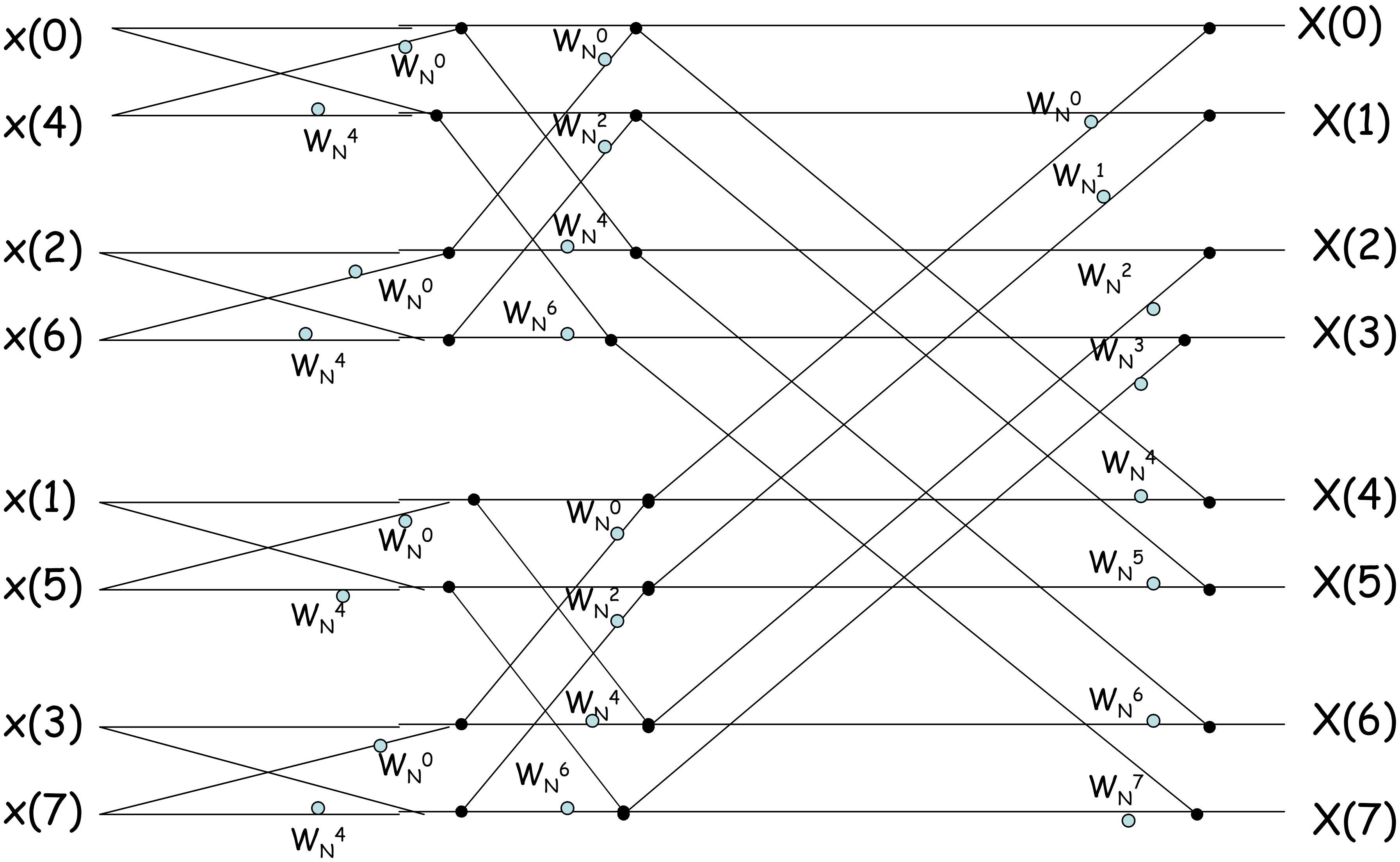

Signal Flow Graph

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No. of Complex Multiplications & Additions:

4 N/4 pt. DFTs : $4(N/4)^2$

$\sim N^2/4$ (Complex Mul. & Add.)

Again Reduction by a factor of 2

Total reduction: by a factor of 4

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Splitting (Decimation) is done until reduced to one point sequence

For $$N = 2^{\gamma}$$

Decimation is performed $$\gamma = \log_2 N$$ times

The total No. of Complex Multiplications & Additions are reduced to

$$N \log_2 N$$

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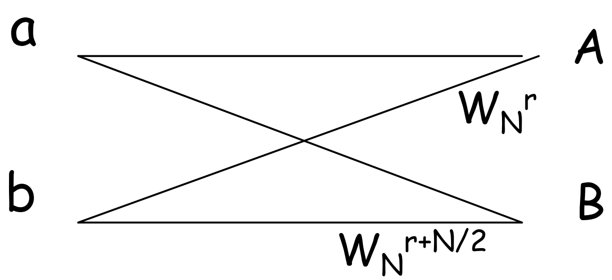
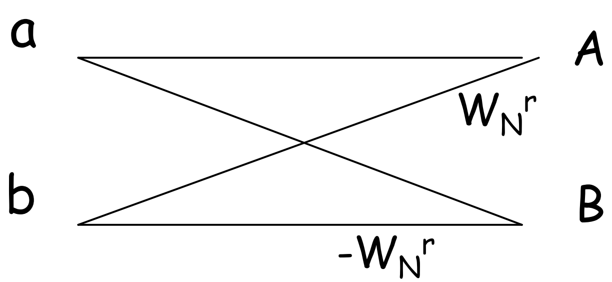
Butterfly Structure



$A = a + W_N^r.b$

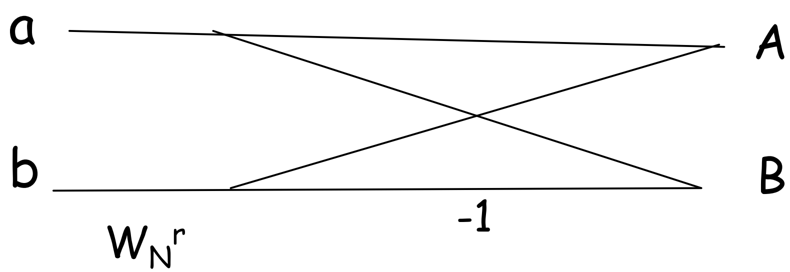
$B = a + W_N^{r+N/2}.b$



Symmetry Property

$A = a + W_N^r.b$

$B = a - W_N^r.b$



$A = a + W_N^r.b$

$B = a - W_N^r.b$

No. of multiplications per butterfly
Is ONE (instead of TWO)

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Hence, total computations are :

Complex Multiplications are : $\frac{N}{2}\log_2 N$

Complex Additions are : $N\log_2 N$

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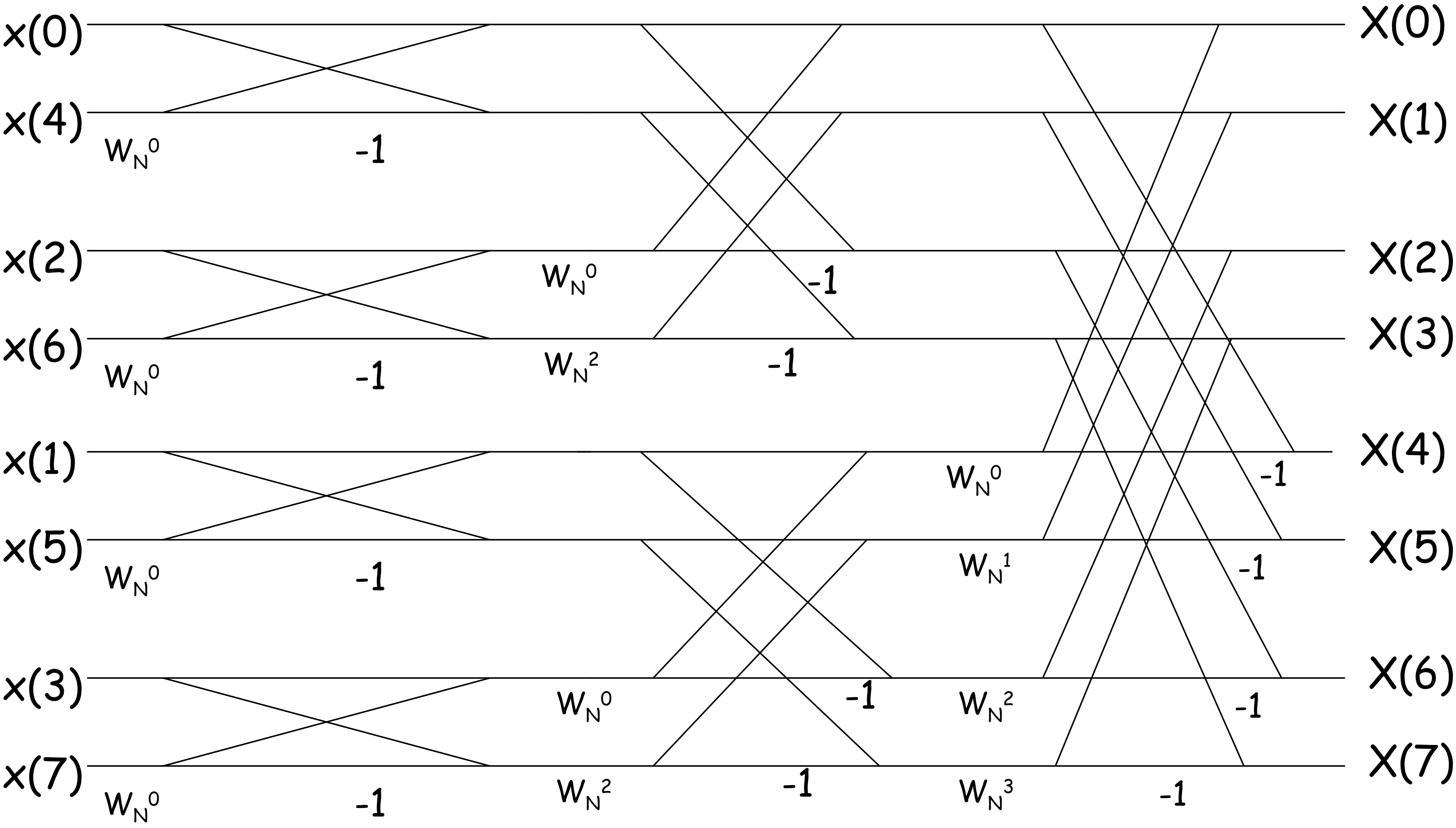

Final Signal Flow Graph of DIT-FFT Algorithm

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Order of input sequence is not natural but definite.

Initial Input sequence	Binary Sequence of index	Bit reversed Sequence	Index of final input sequence	Final Input Sequence
x(0)	000	000	0	x(0)
x(1)	001	100	1	x(4)
x(2)	010	010	2	x(2)
x(3)	011	110	3	x(6)
x(4)	100	001	4	x(1)
x(5)	101	101	5	x(5)
x(6)	110	011	6	x(3)
x(7)	111	111	7	x(7)

Bit Reversal

Input Sequence is bit reversed

16 pt. DFT

Binary Sequence	Input order	Output order
0000	x(0)	X(0)
0001	x(8)	X(1)
0010	x(4)	X(2)
0011	x(12)	X(3)
0100	x(2)	X(4)
0101	x(10)	X(5)
0110	x(6)	X(6)
0111	x(14)	.
1000	x(1)	.
1001	x(9)	.
1010	x(5)	.
1011	x(13)	.
1100	x(3)	.
1101	x(11)	.
1110	x(7)	X(14)
1111	x(15)	X(15)

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In Place Computations :

Butterfly Operation on input pair of complex nos. (a, b) Produces output pair of complex nos. (A, B)

Same locations can be used to store (A, B) i.e. (A, B) is stored in place of (a, b)

Hence same 2N locations are used throughout the computation to store the butterfly result.

Hence computations are done in place.

Input data sequence stored in Bit Reversed Order & Butterfly computations done In Place, Output DFT sequence is obtained in Natural Order.

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DIT - FFT Algorithm:

- Pad input sequence, of N samples, with ZERO's until the number of samples is the nearest power of two. e.g. 12 samples are padded with 4 zeros to make length of input sequence 16 (2^4)
- Bit reverse the input sequence. e.g. x(3) = 0011 goes to 1100 = 12^{th} position in array
- Perform butterfly computations $\gamma = \log_2 N$ stages

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Radix - 2 Decimation in Frequency - FFT algo. : (Radix – 2 DIF – FFT algorithm)

N is considered as integer power of 2

Output sequence X(k) is divided into smaller sequences.

N-point data sequence, x(n), is splitted into
two N/2 point data sequences
containing first half and next half data points
i.e. {x(0), x(1), x(2), x(3)}
{x(4), x(5), x(6), x(7)}

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

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$$X(k)=\sum_{n=0}^{N/2-1}x(n)W_N^{nk}+\sum_{n=N/2}^{N-1}x(n)W_N^{nk}$$

$$X(k)=\sum_{n=0}^{N/2-1}x(n)W_N^{nk}+\sum_{n=0}^{N/2-1}x(n+\frac{N}{2})W_N^{(n+\frac{N}{2})k}$$

$$X(k)=\sum_{n=0}^{N/2-1}x(n)W_N^{nk}+W_N^{\frac{N}{2}k}\sum_{n=0}^{N/2-1}x(n+\frac{N}{2})W_N^{nk}$$

Note: Two summations are not N/2 pt. DFTs

$$W_N^{\frac{N}{2}k}=(-1)^k$$

Periodicity & Symmetry Property

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + (-1)^k \sum_{n=0}^{N/2-1} x(n + \frac{N}{2}) W_N^{nk}$$

Now consider k even & k odd separately

$$X(2r) = \sum_{n=0}^{N/2-1} [x(n) + x(n + \frac{N}{2})] W_N^{2rn}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \{[x(n) - x(n + \frac{N}{2})] W_N^{n}\} W_N^{2rn}$$

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$$W_N^2 = W_{N/2}$$

$$X(2r) = \sum_{n=0}^{N/2-1} [x(n) + x(n + \frac{N}{2}]W_{N/2}^{rn}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \{[x(n) - x(n + \frac{N}{2}]W_N^n\}W_{N/2}^{rn}$$

Above eqs. are two N/2 pt. DFTs

$$X(2r) = \sum_{n=0}^{N/2-1} g(n)W_{N/2}^{rn}$$

$g(n)=x(n)+x(n+N/2)$

$$X(2r+1) = \sum_{n=0}^{N/2-1} h(n)W_N^n W_{N/2}^{rn}$$

$h(n)=x(n)-x(n+N/2)$

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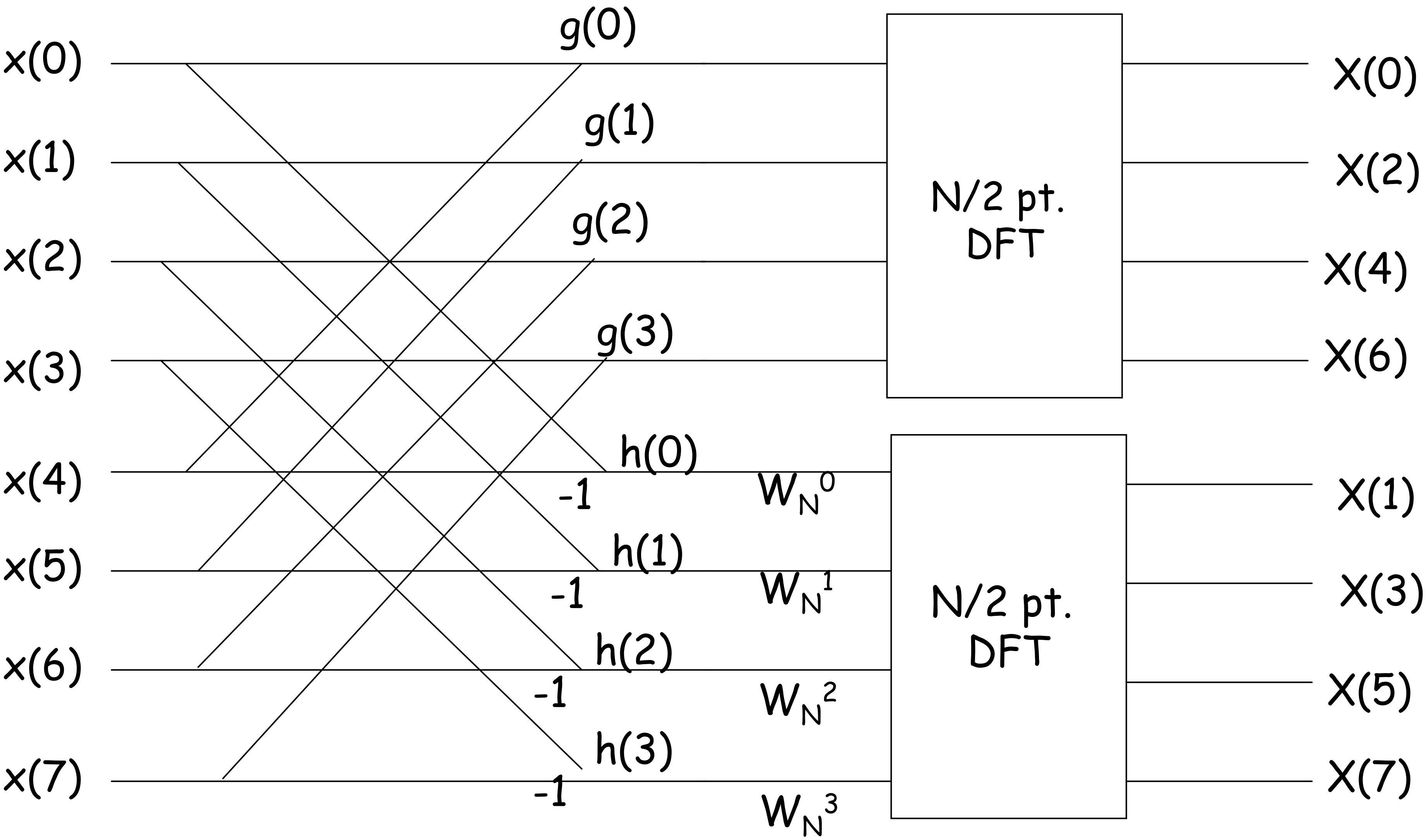

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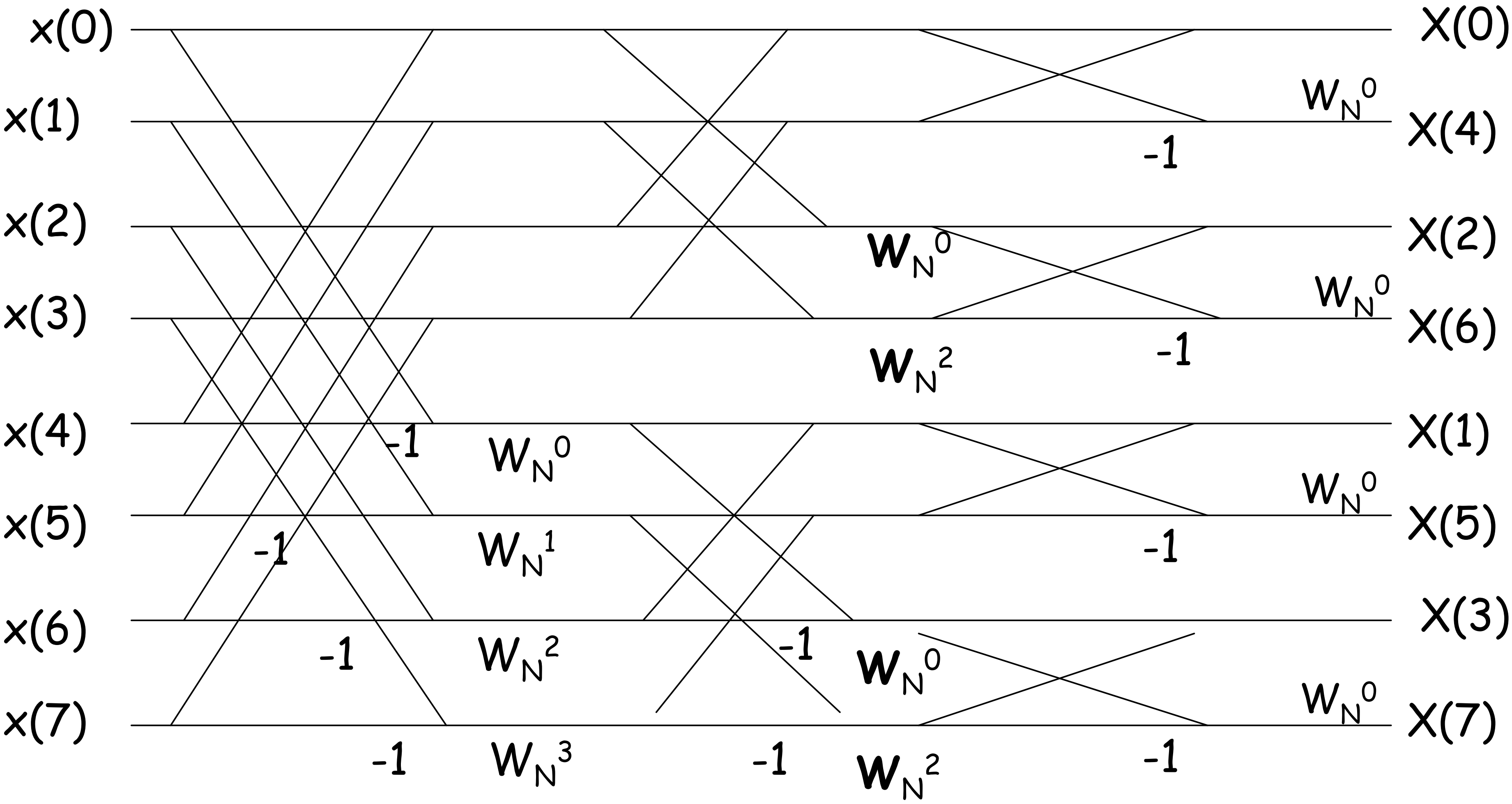

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