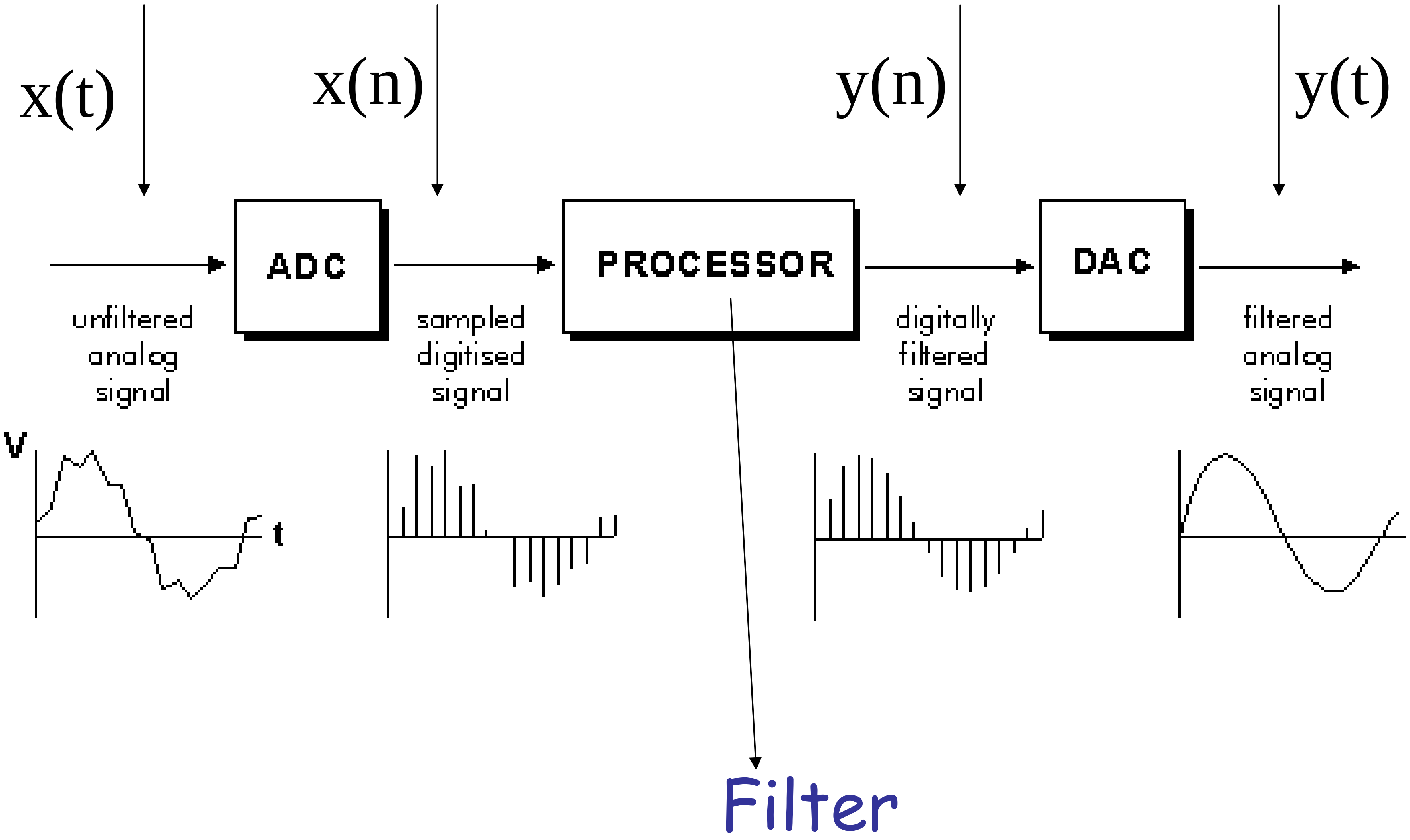
Finite Response Filters
or
FIR Filters

Mrs.Aarti Bang,
VIIT, Pune



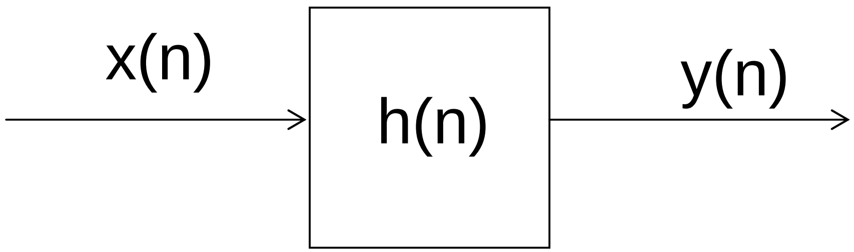
Mrs.Aarti Bang,
VIIT, Pune

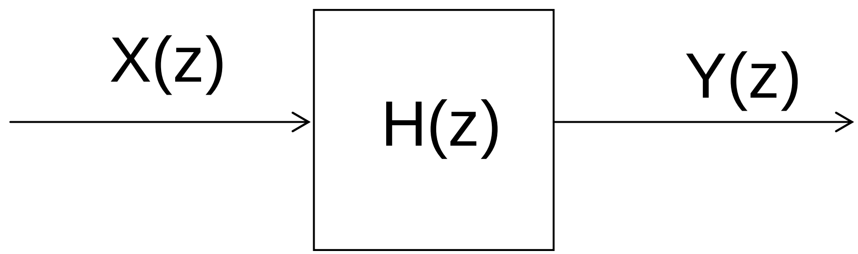
A filter is a frequency selective device

Passes certain band of frequencies and
Blocks frequencies out of this band.

A Digital Filter is a mathematical algorithm that operates on a discrete time signal (x(n), input), to produce a discrete time output signal (y(n), output), to achieve the filtering objective.

The digital filter is characterized by its Impulse Response (h(n))





Block Diagram Representation

Mrs.Aarti Bang,
VIIT, Pune

Digital Systems are characterised by Linear Constant Coefficient Difference equation(LCCDE)

$$y(n)=\sum_{k=0}^{M} b_k x(n-k)-\sum_{k=1}^{N} a_k y(n-k)$$

$$y(n)=b_0 x(n)+b_1 x(n-1)+........+b_M x(n-M)$$
$$-[a_1 y(n-1)+a_2 y(n-2)+......+a_N y(n-N)]$$

Output depends on present and past inputs and past outputs

------------- Infinite Impulse Response (IIR) Systems

Mrs.Aarti Bang,
VIIT, Pune

Output depends on present and past inputs only
------------- Finite Impulse Response (FIR) Systems

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = b_0 x(n) + b_1 x(n-1) + ... + b_M x(n-M)$$

$$y(n) = h_0 x(n) + h_1 x(n-1) + ... + h_M x(n-M)$$

Mrs.Aarti Bang,
VIIT, Pune

Alternative Representation is (in Z - Domain)

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

IIR System

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

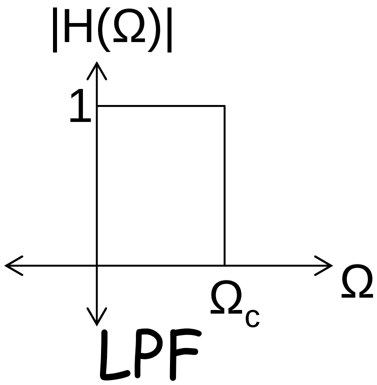
FIR System

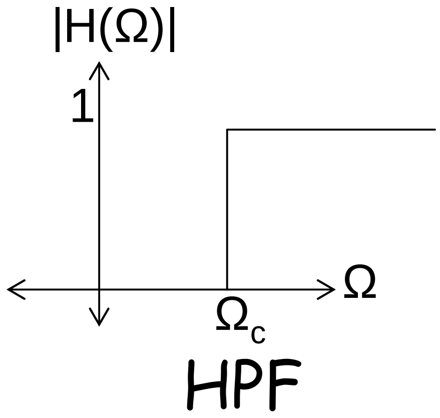
$$H(z) = \sum_{k=0}^{M} h_k z^{-k}$$

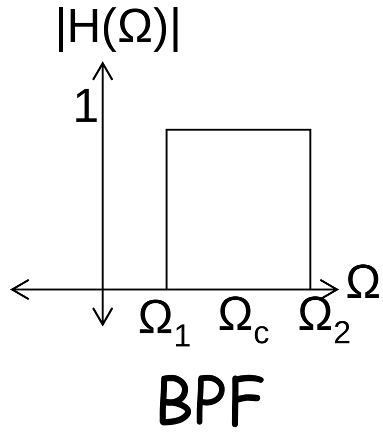
Mrs.Aarti Bang,
VIIT, Pune

Filter Characteristics

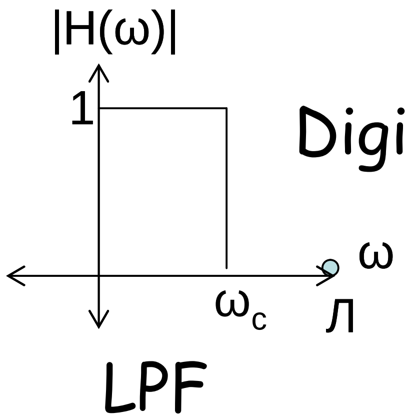
Brick wall







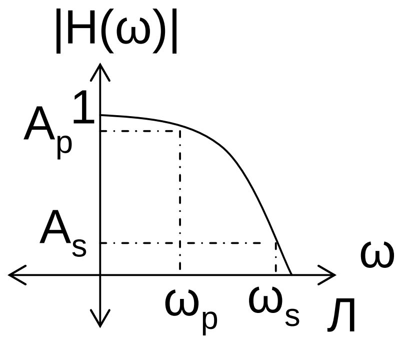
Analog



Digital

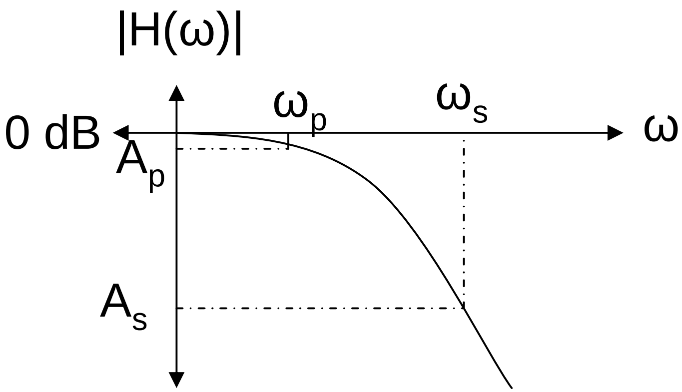
Mrs.Aarti Bang,
VIIT, Pune

Practical Characrteristics



A_p is minimum pass band gain
A_s is maximum stop band gain
ω_p is the pass band edge frequency
ω_s is the stop band edge frequency

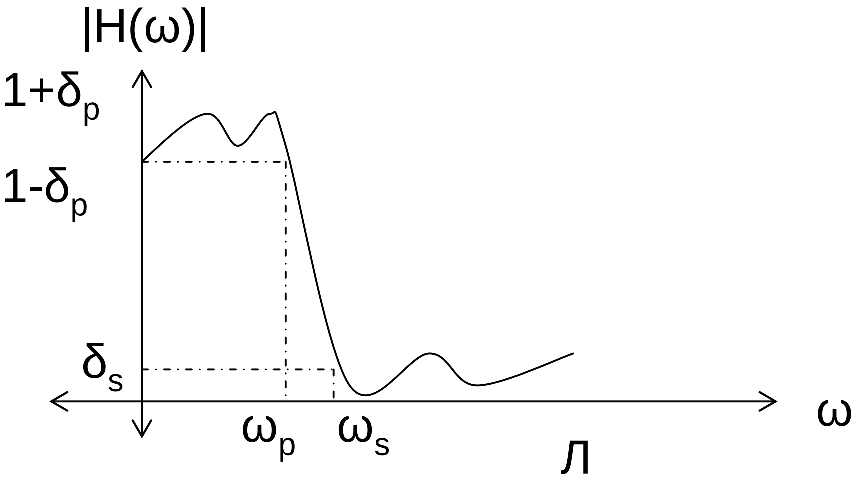
Gain Characteristics



A_p in dB is maximum pass band attenuation
A_s in dB is minimum stop band attenuation
ω_p is the pass band edge frequency
ω_s is the stop band edge frequency

Attenuation Characteristics

Mrs.Aarti Bang,
VIIT, Pune



$\delta_p \rightarrow$ pass band ripple

$\delta_s \rightarrow$ stop band ripple

$$A_p = 1 - \delta_p$$

$$A_p = -20\log_{10}(1 - \delta_p)db$$

$$A_s = -20\log_{10}(\delta_s)db$$

Mrs.Aarti Bang,
VIIT, Pune

Order of Filter : (N or M)

The rate at which gain decreases from pass band to stop band, along the frequency.

Mrs.Aarti Bang,
VIIT, Pune

Ideal filters have

- Constant Magnitude characteristics
- Linear Phase characteristics

within the pass band

Mrs.Aarti Bang,
VIIT, Pune

A filter has linear phase response if

$$y(n) = cx(n-n_0)$$

i.e. output is simply delayed and amplitude scaled version of input.

Linear phase characteristics can be written as

$\Theta(\omega) = -\omega n_0$ (negative sign is for phase lag)

$$\frac{d\Theta(\omega)}{d\omega} = -n_0 = \tau_g(\omega)$$

Group delay or Envelope delay that is constant

i.e. all frequency components of the input signal undergo same (constant) time delay.

Mrs.Aarti Bang,
VIIT, Pune

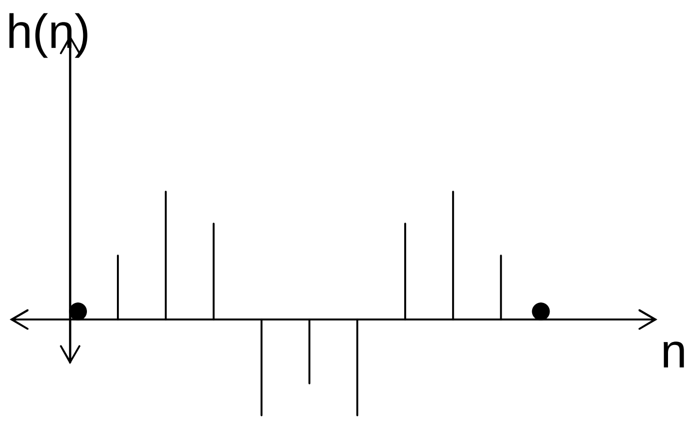
FIR filters have linear phase response if their impulse response is symmetric or antisymmetric

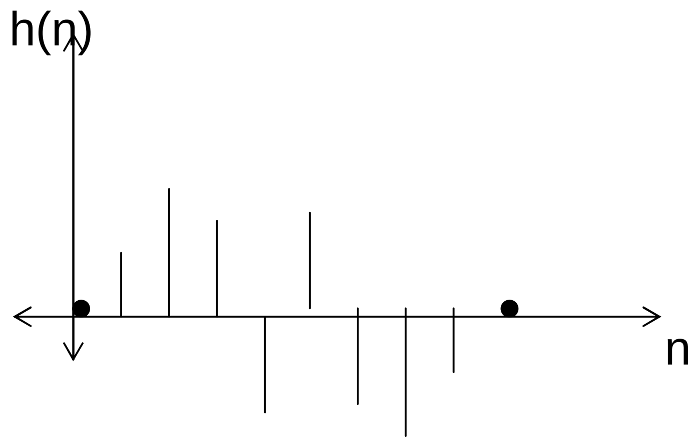
$$h(n) = h(M-1-n)$$

Here M=11

$$h(n) = -h(M-1-n)$$

Here M=10





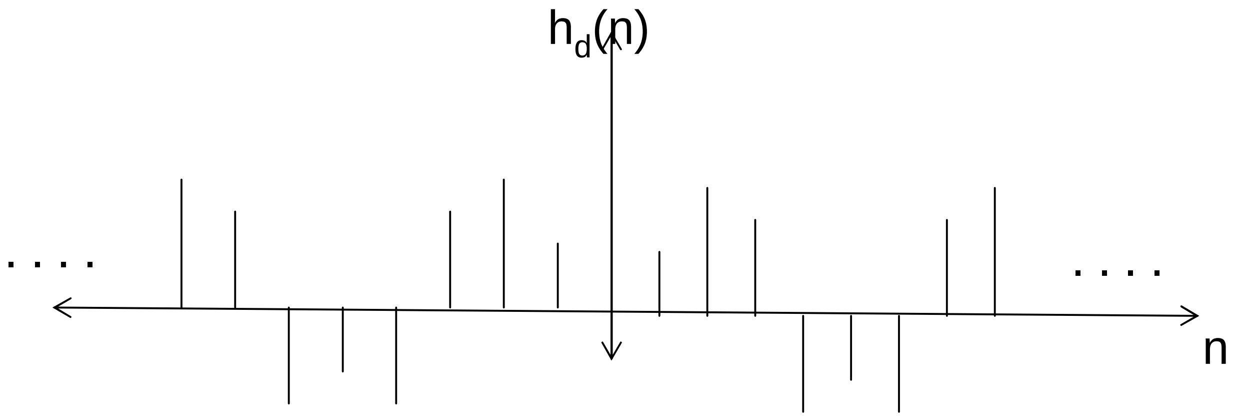
Mrs.Aarti Bang,
VIIT, Pune

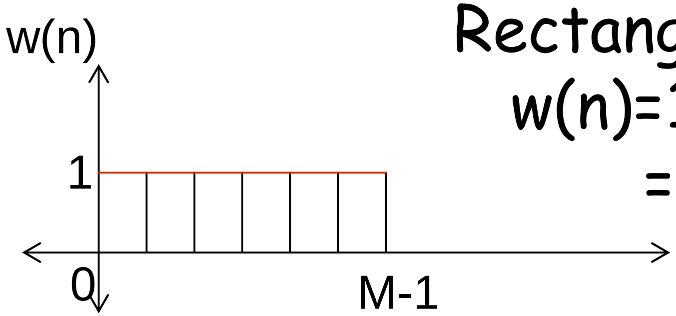
Characteristics of FIR filters :

Have **linear phase response**
Are inherently **stable**
Are **all zero** systems
Need **higher order** for similar magnitude

Design of FIR filters means,
design of unit sample response h(n), means
design of M coefficients h(n), n=0,1, ….,M-1 from a desired frequency response $H_d(\omega)$

Mrs.Aarti Bang,
VIIT, Pune



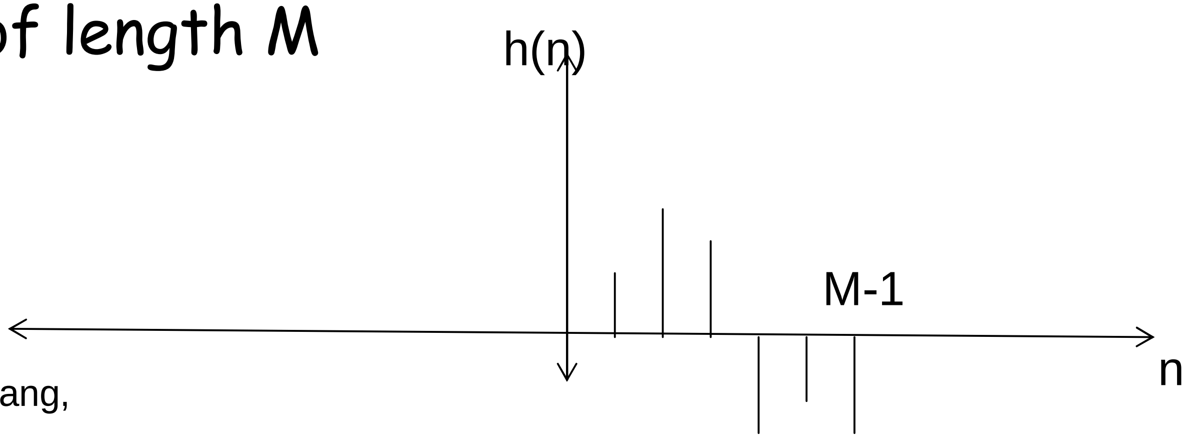


Rectangular window

$w(n)=1 \quad n=0,1,\ldots,M-1$

$\quad\quad =0 \quad \text{otherwise}$

This is equivalent of multiplying $h_d(n)$ by a window sequence w(n) of length M



Mrs.Aarti Bang,
VIIT, Pune

Methods for designing FIR filters

- Window method
- Frequency Sampling method

Mrs.Aarti Bang,
VIIT, Pune

Design of FIR filters using Window method

$H_d(\omega) \rightarrow$ desired frequency response of digital filter

$h_d(n) \rightarrow$ corresponding impulse/unit sample response

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

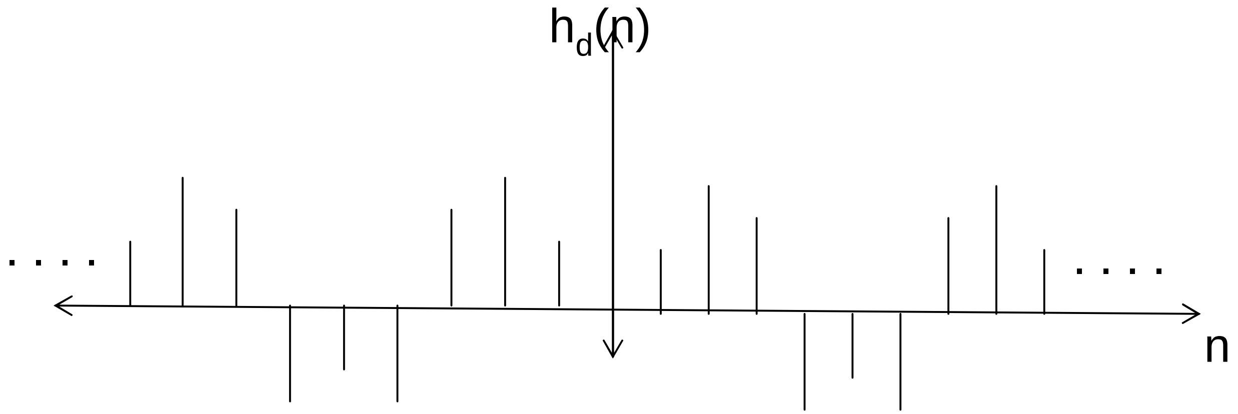
$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{-j\omega n} d\omega$$

Discrete Time Fourier Transform pair eq.

Mrs.Aarti Bang,
VIIT, Pune

The unit sample response $h_d(n)$, obtained, is Infinite in duration and Non-causal

→Non-realizable



$h_d(n)$ should be made finite and causal.
i.e. truncated to length M.

Mrs.Aarti Bang,
VIIT, Pune

Steps in designing FIR filter using window method

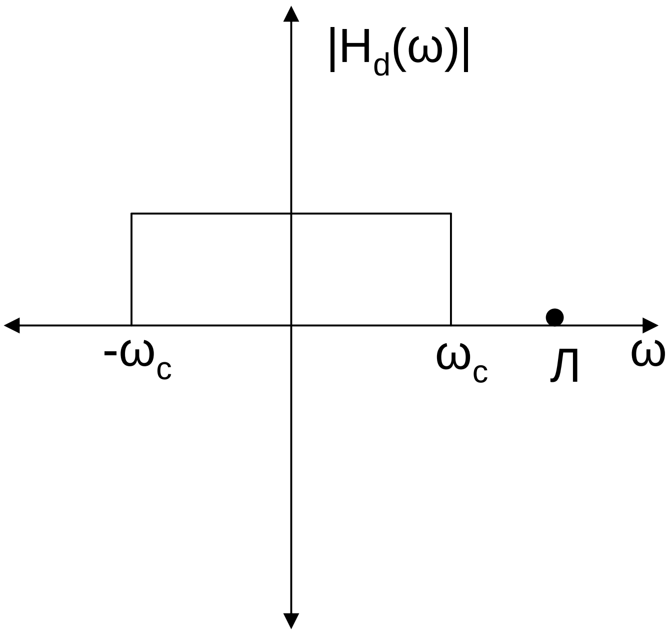
- Given $H_d(\omega)$
- Determine $h_d(n)$, using Inverse Fourier Transform
- Compute $h(n) = h_d(n) \times w(n) \quad n=0,1,\ldots\ldots,M-1$

Specifications given are:

$$H_d(\omega) = 1 \qquad |\omega| <= \omega_c$$
$$= 0 \qquad \omega_c <= |\omega| <= \pi$$

$$H_d(\omega) = e^{-j\omega\tau} \qquad |\omega| <= \omega_c$$
$$= 0 \qquad \omega_c <= |\omega| <= \pi$$

where $\tau = (M-1)/2$



Mrs.Aarti Bang,
VIIT, Pune

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega$$

$$h_d(n) = \frac{1}{(n-\tau)\pi} [\sin\omega_c(n-\tau)]$$

for n not equal τ

$$h_d(n) = \frac{\omega_c}{\pi}$$

for n = τ

Mrs.Aarti Bang,
VIIT, Pune

What is τ ?

If filter is symmetric,

$h_d(n) = h_d(M-1-n)$

$$\frac{\sin\omega_c(n-\tau)}{(n-\tau)\pi} = \frac{\sin\omega_c(M-1-n-\tau)}{(M-1-n-\tau)\pi}$$

$-(n-\tau) = M-1-n-\tau$

$\tau = (M-1)/2$

$$h_d(n) = \frac{\sin\omega_c(n-\frac{M-1}{2})}{(n-\frac{M-1}{2})\pi}$$

for n not equal τ

$$h_d(n) = \frac{\omega_c}{\pi}$$

for $n = \tau$

Mrs.Aarti Bang,
VIIT, Pune

Effect of Rectangular window function on desired frequency response

w(n) = 1 n=0,1,….,M-1
= 0 otherwise

Impulse response of FIR filter is
$h(n) = h_d(n) \times w(n)$

Multiplication in time domain is equivalent to Convolution in frequency domain.

$$H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) W(\omega - v)\,dv$$

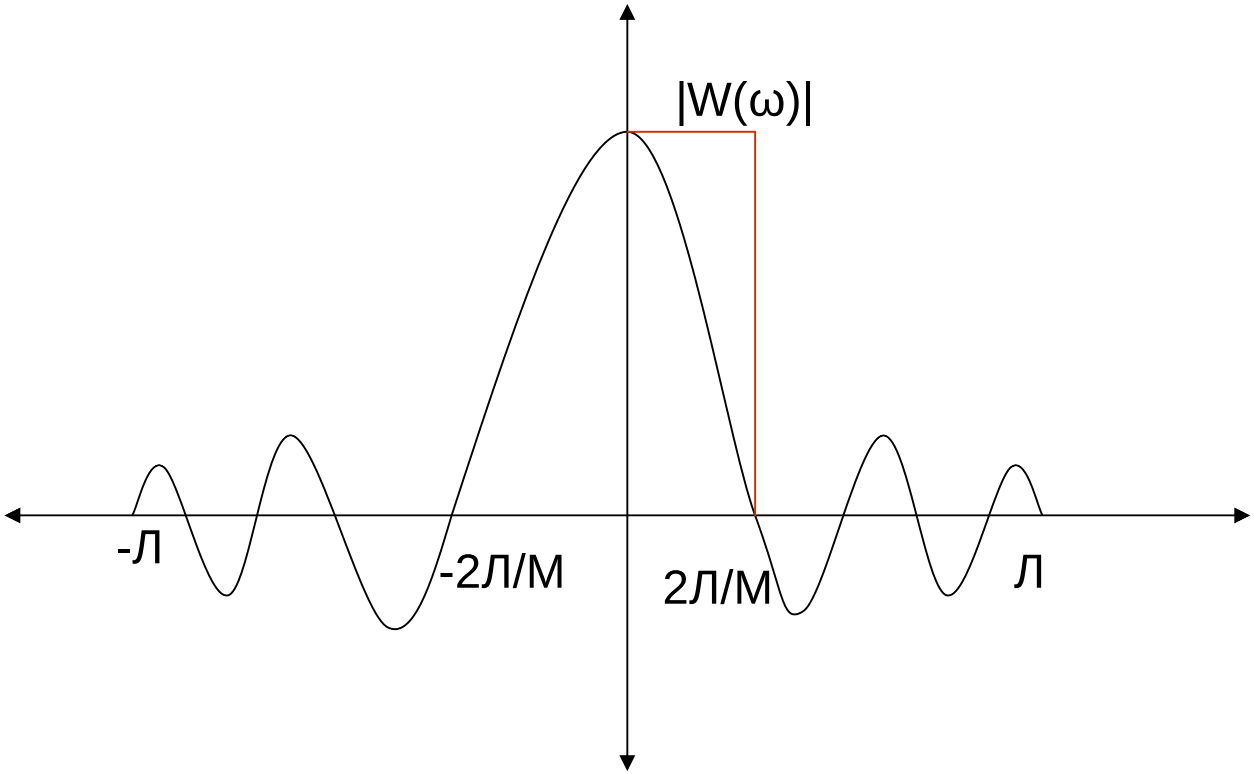
Mrs.Aarti Bang,
VIIT, Pune

Fourier Transform of window function is

$$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n}$$

$$W(\omega) = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = e^{-j\omega\frac{M-1}{2}} \frac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}$$

Mrs.Aarti Bang,
VIIT, Pune



Frequency response of rectangular window

- There is main lobe, contains most of the energy, width is 4 Л/M
- There are a no. of side lobes that decay rapidly.
- As M increases, width of main lobe becomes narrow.
- Height of each side lobe increases as M increases
- Width of each side lobe decreases as M increases.
- Area of each side lobe remains unchanged.

Mrs. Aarti Berug,
VIIT, Pune

- The truncation of infinite series results in undesirable oscillations in the pass band and stop band.
- These oscillations are due to abrupt discontinuity at the edge of window.
- These oscillations can be reduced using a tapered window.

- The oscillatory behaviour near the band edge of filter is called Gibbs phenomenon.

Mrs.Aarti Bang,
VIIT, Pune

Desirable characteristics of window function :

The Fourier Transform/Frequency response of window function should have a small width main lobe containing as much as the total energy as possible.

It should have side lobes that decrease in energy rapidly as ω tends to π.

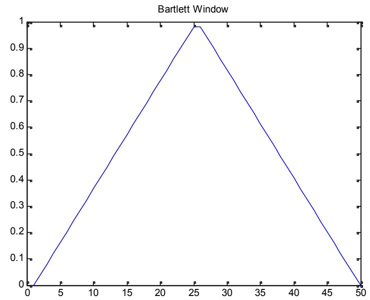
Mrs.Aarti Bang,
VIIT, Pune

Window Functions

w(n), 0<=n<=M-1

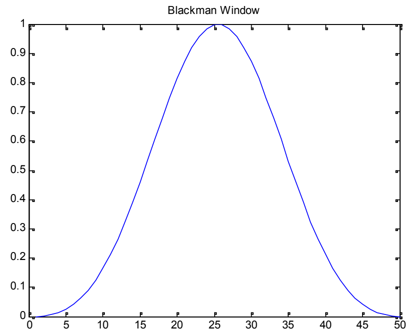
Bartlett (Triangular)

$$1-\frac{2\left|n-\frac{M-1}{2}\right|}{M-1}$$



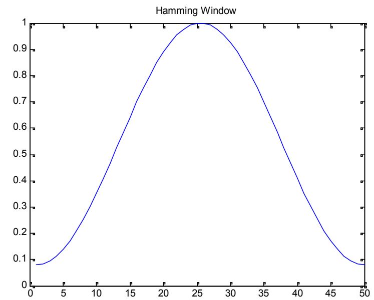
Blackman

$$0.42-0.5\cos\frac{2\pi n}{M-1}+0.08\cos\frac{4\pi n}{M-1}$$



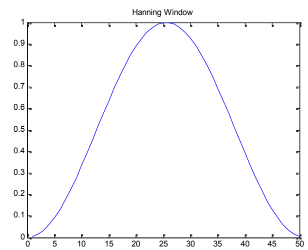
Hamming

$$0.54-0.46\cos\frac{2\pi n}{M-1}$$

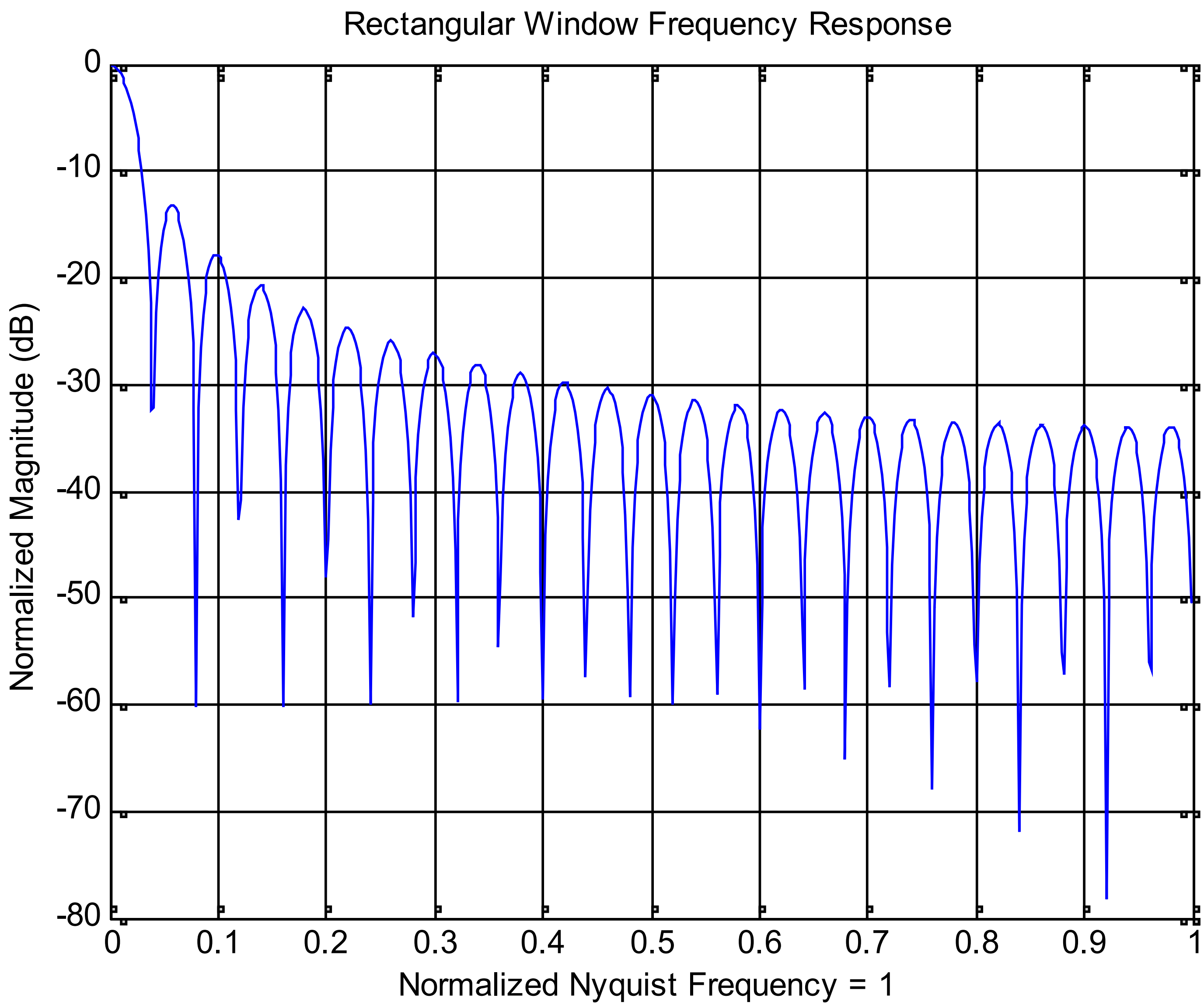


Hanning

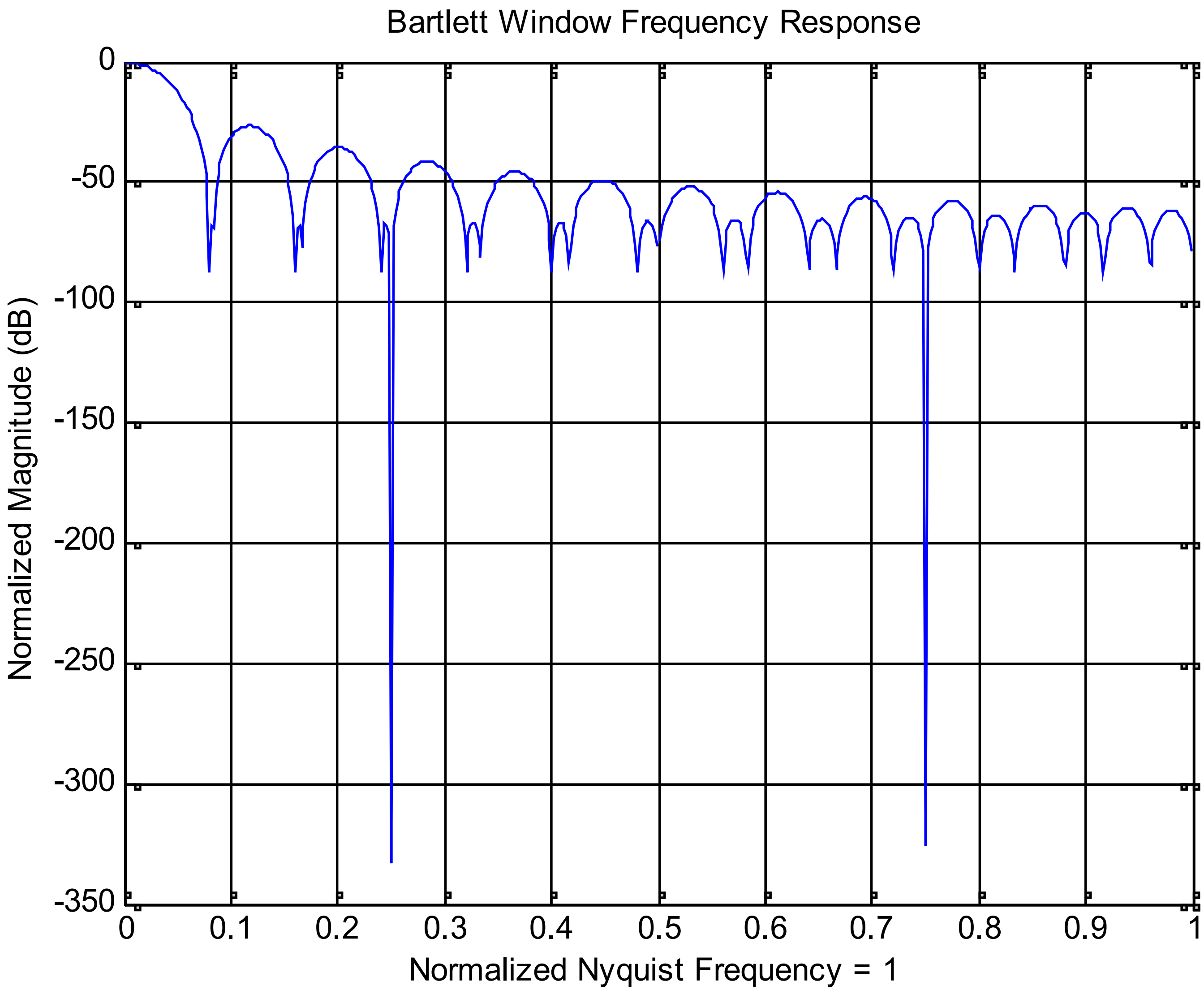
$$\frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right)$$



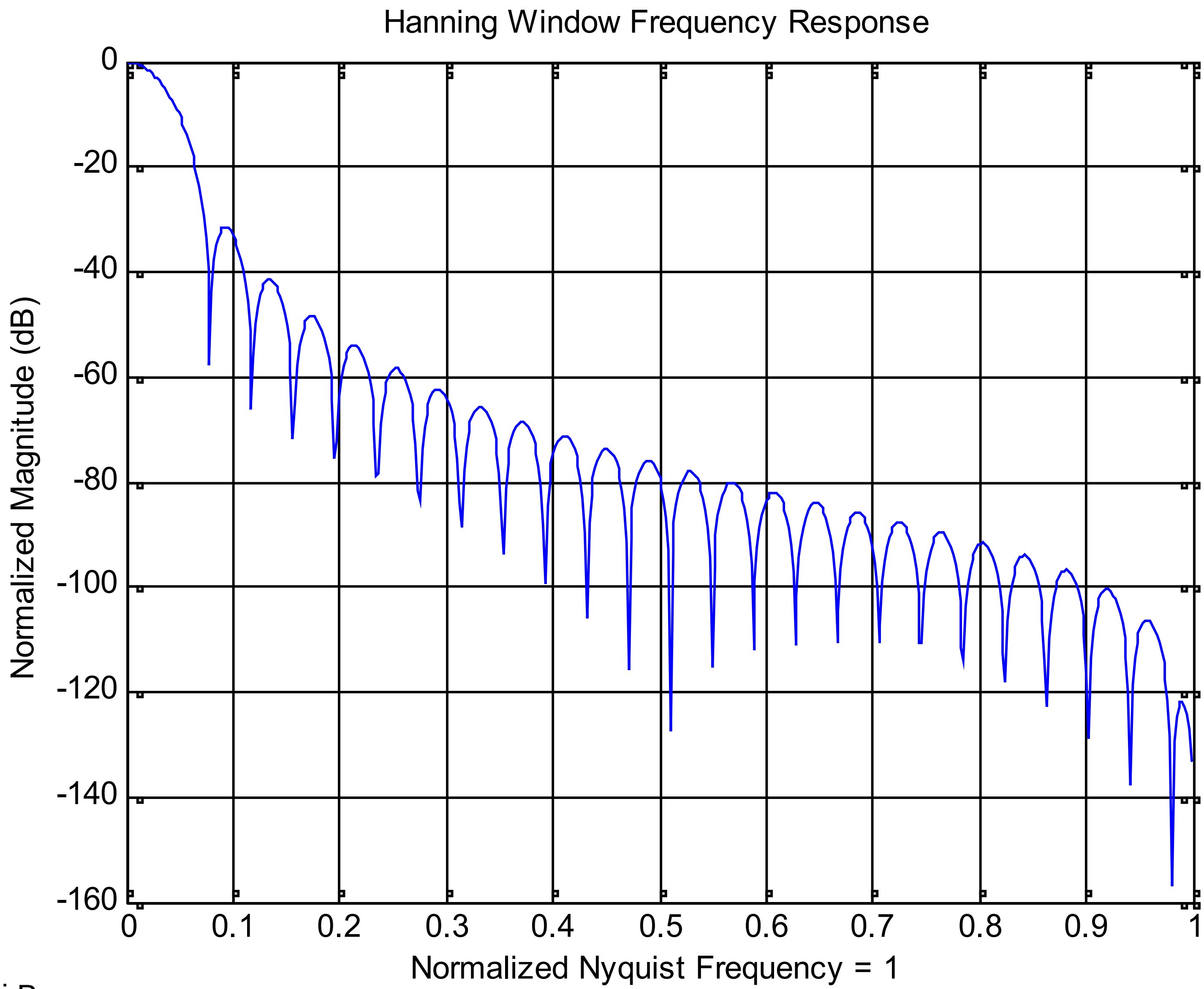
Mrs.Aarti Bang,
VIIT, Pune



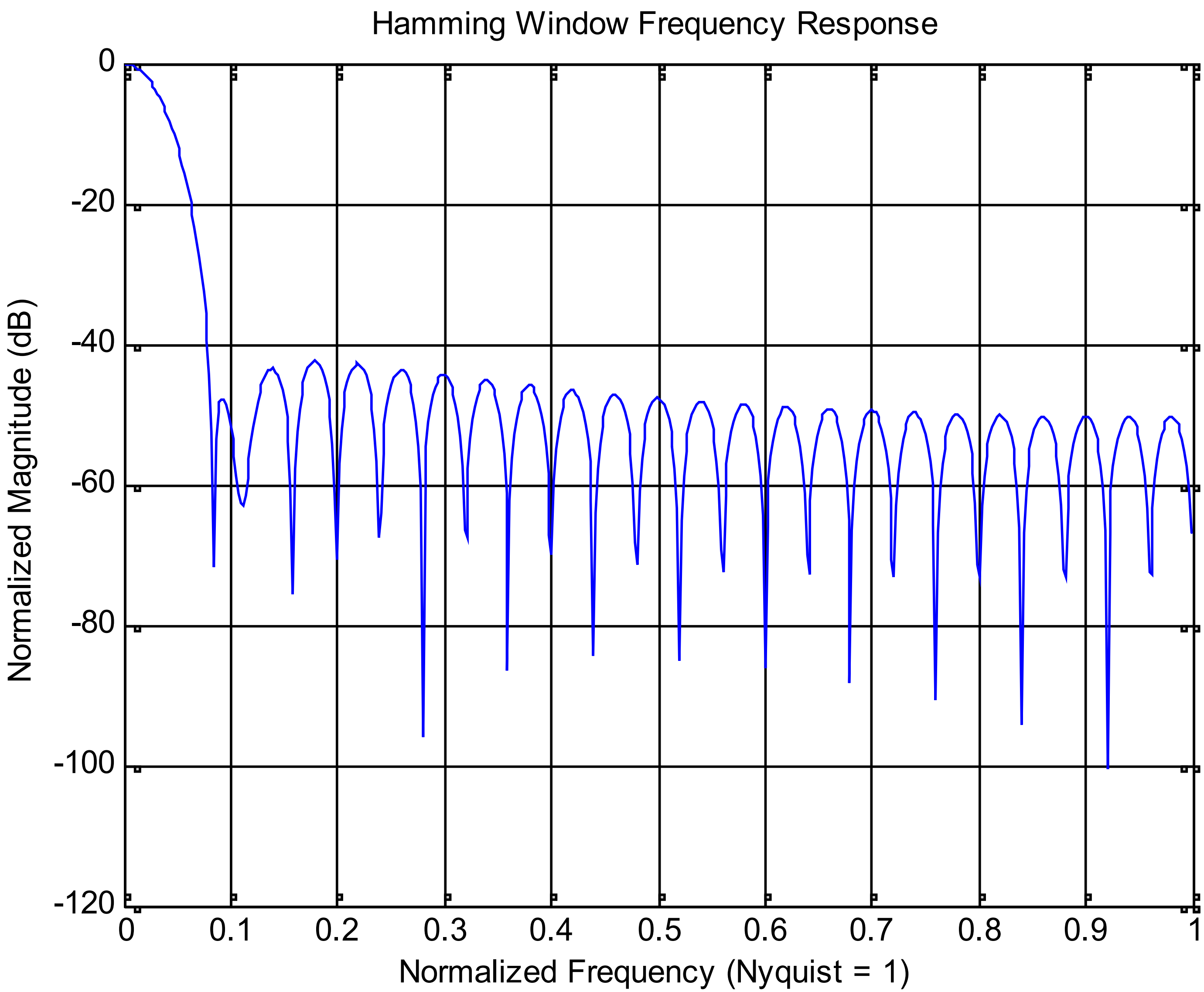
Mrs.Aarti Bang,
VIIT, Pune



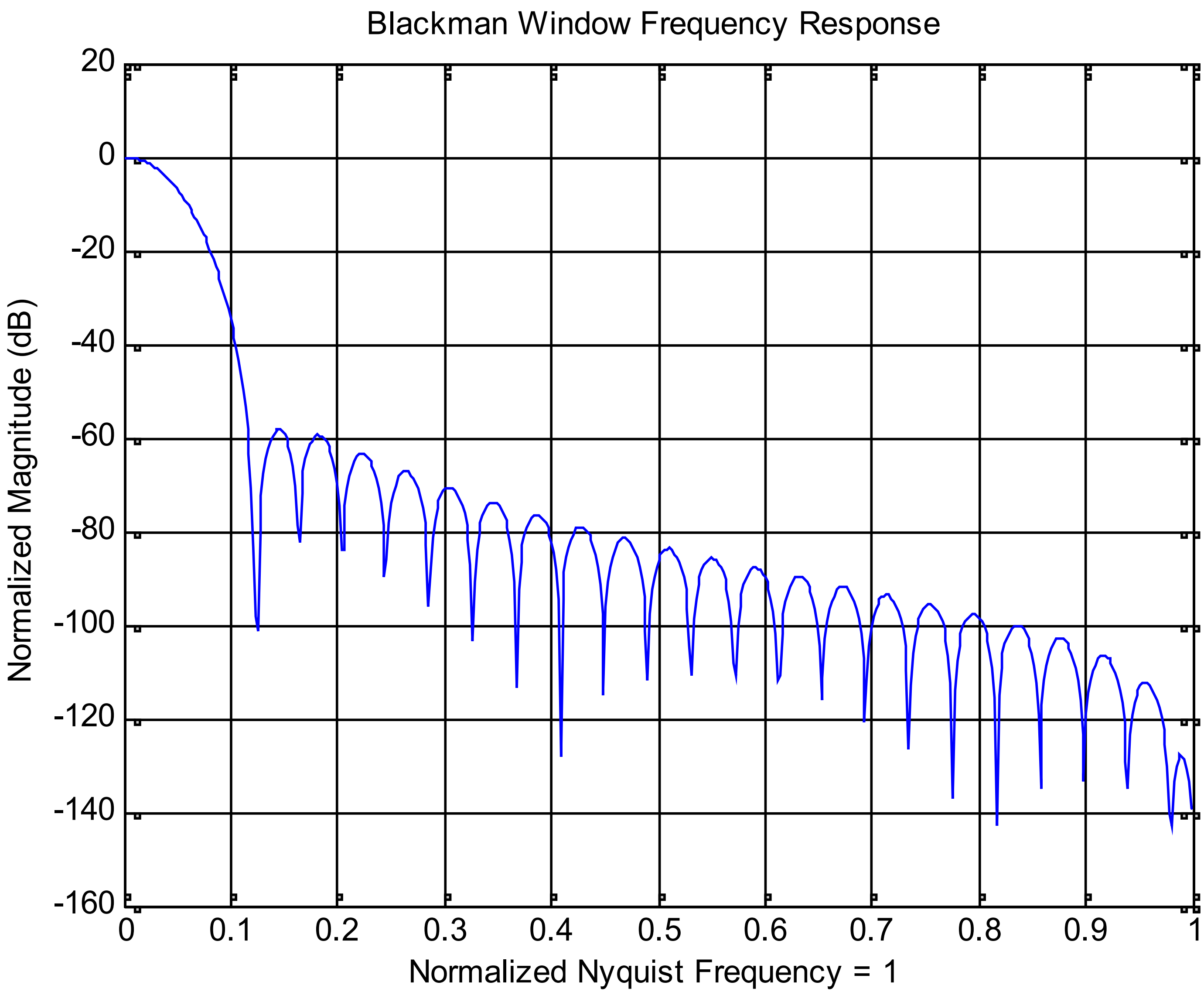
Mrs.Aarti Bang,
VIIT, Pune



Mrs.Aarti Bang,
VIIT, Pune



Mrs.Aarti Bang,
VIIT, Pune



Mrs.Aarti Bang,
VIIT, Pune

Type of Window	Width of Main Lobe	Min. Stopband Attenuation (db)	Peak side Lobe (db)	Normalized Transition width Δf
Rectangular	4Л/M	-21	-13	0.9/M
Bartlett	8Л/M	-25	-27	
Hanning	8Л/M	-44	-32	3.1/M
Hamming	8Л/M	-53	-43	3.3/M
Blackman	12Л/M	-74	-58	5.5/M

Window Functions :

- Minimize the ringing effects at the band edge.
- Result in lower side lobes.
- At the expense of increase in the width of transition band.

Mrs.Aarti Bang,
VIIT, Pune

Estimation of Filter Order (M) :

Kaiser's formula:

$$M = \frac{-10\log_{10}(\delta_p \delta_s) - 13}{14.6\Delta f}$$

or

$$M = \frac{-10\log_{10}(\delta_p \delta_s) - 13}{14.6(\omega_{st} - \omega_p P / 2\pi}$$

M is inversely proportional to transition bandwidth ($\omega_s - \omega_p$) and not on transition band location.

Mrs.Aarti Bang,
VIIT, Pune

Estimation of Filter Order (M) :

- Hermann-Rabiner-Chan's Formula:

$$M \cong \frac{D_\infty(\delta_p,\delta_s) - F(\delta_p,\delta_s)[(\omega_s-\omega_p)/2\pi]^2}{(\omega_s-\omega_p)/2\pi}$$

where

$$D_\infty(\delta_p,\delta_s) = [a_1(\log_{10}\delta_p)^2 + a_2(\log_{10}\delta_p) + a_3]\log_{10}\delta_s$$
$$+[a_4(\log_{10}\delta_p)^2 + a_5(\log_{10}\delta_p) + a_6]$$

$$F(\delta_p,\delta_s) = b_1 + b_2[\log_{10}\delta_p - \log_{10}\delta_s]$$

with

$$a_1 = 0.005309,\ a_2 = 0.07114,\ a_3 = -0.4761$$
$$a_4 = 0.00266,\ a_5 = 0.5941,\ a_6 = 0.4278$$
$$b_1 = 11.01217,\quad b_2 = 0.51244$$

Mrs.Aarti Bang,
VIIT, Pune

FIR Digital Filter Order Estimation

- Formula valid for $\delta_p \geq \delta_s$
- For $\delta_p < \delta_s$, formula to be used is obtained by interchanging δ_p and δ_s
- Both formulae provide only an estimate of the required filter order N
- If specifications are not met, increase filter order until they are met

Mrs.Aarti Bang,
VIIT, Pune

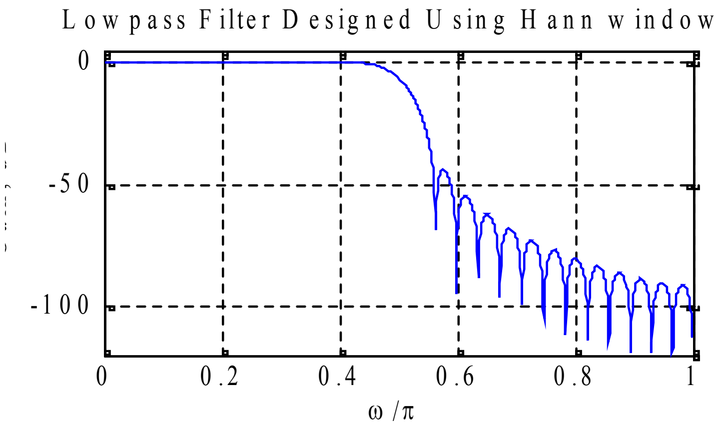
Steps in designing FIR filter

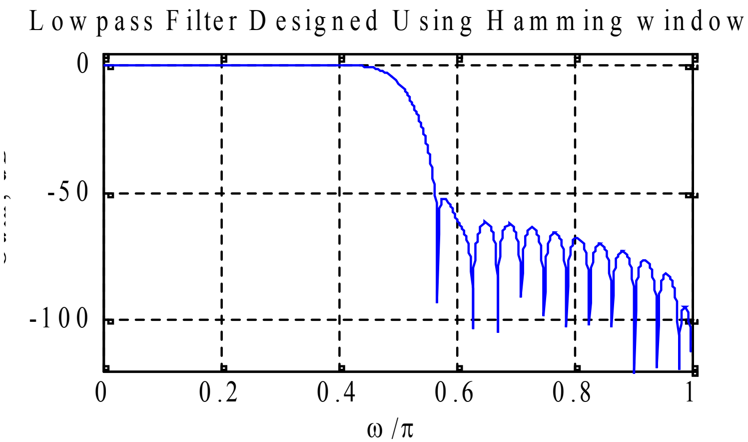
- Specify the desired frequency response $H_d(\omega)$
- Select a window function that satisfies the attenuation specification
- Determine the no. of filter coefficients (M) using the relationship between M & Δf
- Obtain impulse response $h_d(n)$ by evaluating inverse fourier transform
- Obtain M values of chosen window function, $w(n)$
- Obtain actual FIR coefficients, $h(n)$, using the relation $h(n) = h_d(n) \times w(n)$

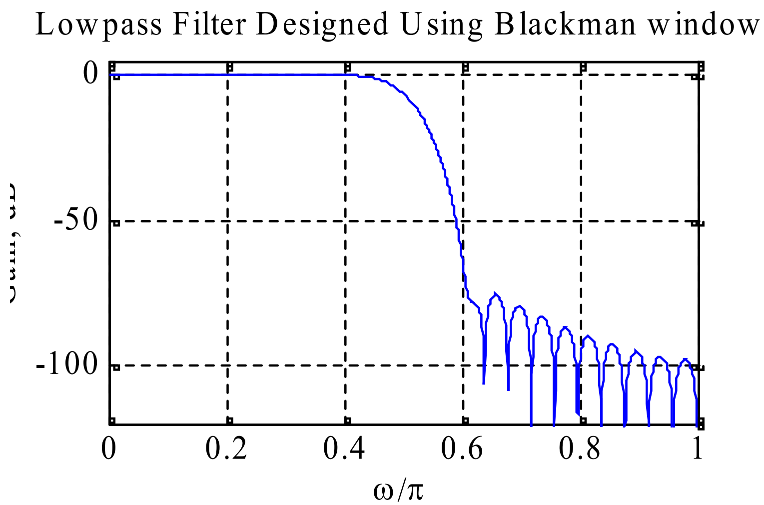
Mrs.Aarti Bang,
VIIT, Pune

Example

- Lowpass filter of length 51 and $\omega_c = \pi / 2$







Mrs.Aarti Bang,
VIIT, Pune

Kaiser Window

$$w(n) = \frac{I_0\left[\beta\left\{\sqrt{1-\left(\frac{2n}{M-1}\right)^2}\right\}\right]}{I_0[\beta]} \qquad -(M-1)/2 \le n \le (M-1)/2$$

$$= 0 \qquad \text{elsewhere}$$

- where $I_0(x)$ is the zero-order modified Bessel function of the first kind.
- β & M controls the way window function tapers at the edges.
- β & M controls the ripple, the stopband attenuation & transition width
- β & M is determined by stopband attenuation requirements.

$$\delta = \min(\delta_p , \delta_s) \qquad A = -20\log_{10}\delta$$

$\beta = 0$ $\qquad A \le 21$ dB

$\beta = 0.5842(A - 21)0.4 + 0.07886(A - 21)$ $\qquad 21 \text{ dB} < A < 50$ dB

$\beta = 0.1102(A - 8.7)$ $\qquad A \ge 50$ dB

Length of filter (Kaiser window) is given by

$$M \geq \frac{\delta_s - 7.95}{14.36 \Delta f}$$

where Δf is the normalized transition width

Mrs.Aarti Bang,
VIIT, Pune

Example: Kaiser Window Design of a Low pass Filter

- Specifications $\omega_p = 0.4\pi, \omega_p = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001$
- Window design methods assume $\delta_1 = \delta_2 = 0.001$
- Determine cut-off frequency
  - Due to the symmetry we can choose it to be $\omega_c = 0.5\pi$
- Compute

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi \qquad A = -20\log_{10}\delta = 60$$

- And Kaiser window parameters

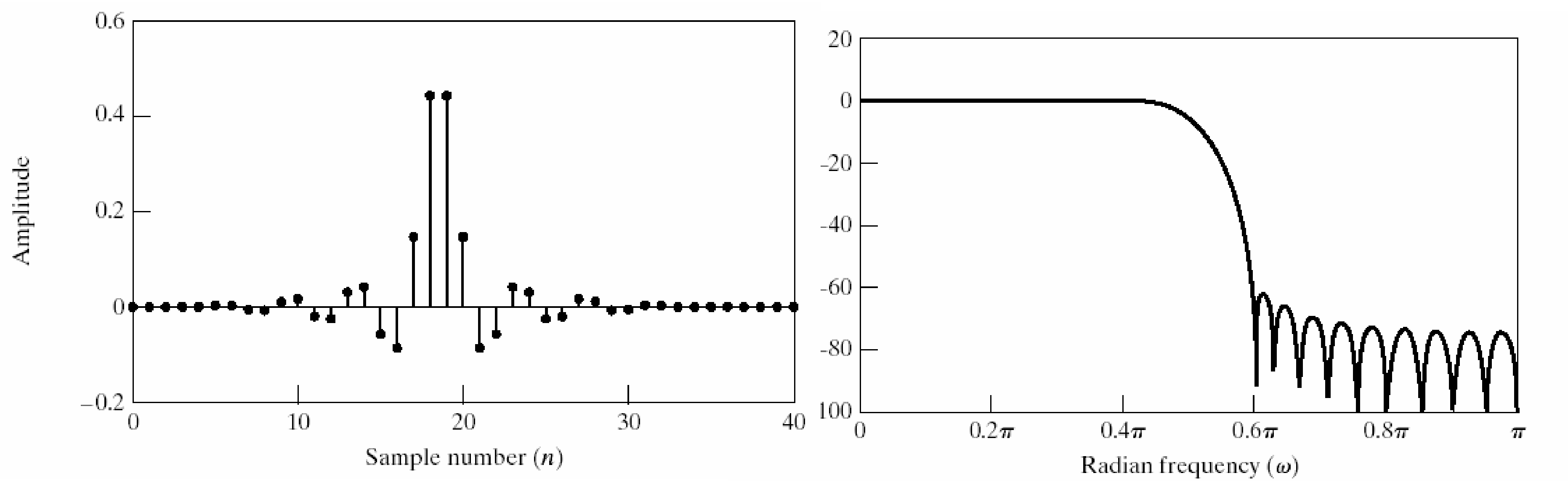
$$\beta = 5.653 \qquad M = 37$$

- Then the impulse response is given as

$$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-18.5)]}{\pi(n-18.5)} \dfrac{I_0\left[5.653\sqrt{1-\left(\dfrac{n-18.5}{18.5}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Mrs.Aarti Bang,
VIIT, Pune

Example Cont’d



Mrs.Aarti Bang,
VIIT, Pune