IIR Filters

- We will consider IIR filters with a rational transfer function (M≤N):

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}}$$

- Design algorithms for continuous-time filters are well known, while Nyquist issues make discrete-time IIR filter design somewhat complicated
- Concept: use a continuous-time filter design, then transform s-plane poles and zeros to the z-plane

- Mapping function must:
  - Take $j\omega$ axis in s-plane and map it to unit circle in z-plane
  - Take left half of s-plane and map to interior of unit circle *(stable analog filter will be converted to a stable digital filter) or (stable poles remain stable poles)*

- Analogue filters have transfer functions:

$$H_a(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N}{b_0 + b_1 s + b_2 s^2 + \ldots + b_M s^M}$$

- Replace s by jw for frequency-response.

- $H_a(s)$ is Laplace Transform of $h_a(t)$.
- In terms of poles and zeros:

$$H_a(s) = K \frac{(s - z_1)(z - z_2)\ldots(s - z_N)}{(s - p_1)(z - p_2)\ldots(s - p_M)}$$

- Many known techniques for deriving $H_a(s)$;
  e.g. Butterworth low-pass approximation

IIR filter Design

- *Transformation of Analog Filter to Digital Filter*

Properties of the transformation

- Must preserve essential properties of the analog frequency response
- Must preserve stability (a stable analog transfer function to be transformed to stable digital filter)

Two important and widely mapping or transformation functions used are:

- Impulse Invariance Transformation
- Bilinear Transformation

Impulse Invariance Transformation

Impulse response of digital filter is obtained by uniformly sampling the impulse response of analog filter. $h(n) = h_a(nT) = h_a(t)$

s – plane to z – plane mapping :

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_k T} z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Here $$H(z) = \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

Poles of H(z) are located at $z_k = e^{p_k T}$ k=0,1,….,N

Analog pole $s = p_k$ → Digital pole $z = e^{p_k T}$

$$\therefore z = e^{sT}$$

We have $s = \sigma + j\Omega$ $z = re^{j\omega}$

$$\therefore re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

$$\therefore r = e^{\sigma T}$$ $$\omega = \Omega T$$

$$\therefore r = e^{\sigma T} \qquad \omega = \Omega T$$

$$\begin{aligned} \therefore if \quad & \sigma < 0 \;\; then \quad r < 1 \quad or \quad |z| < 1 \\ & \sigma > 0 \;\; then \quad r > 1 \quad or \quad |z| > 1 \\ & \sigma = 0 \;\; then \quad r = 1 \quad or \quad |z| = 1 \end{aligned}$$

- Left half poles in s-plane to poles of a digital filter inside unit circle
- Right half poles in s-plane to poles of a digital filter outside unit circle

Drawback of Impulse Invariance Method

The mapping of Ω to ω is not one to one

Here $\omega = \Omega T$

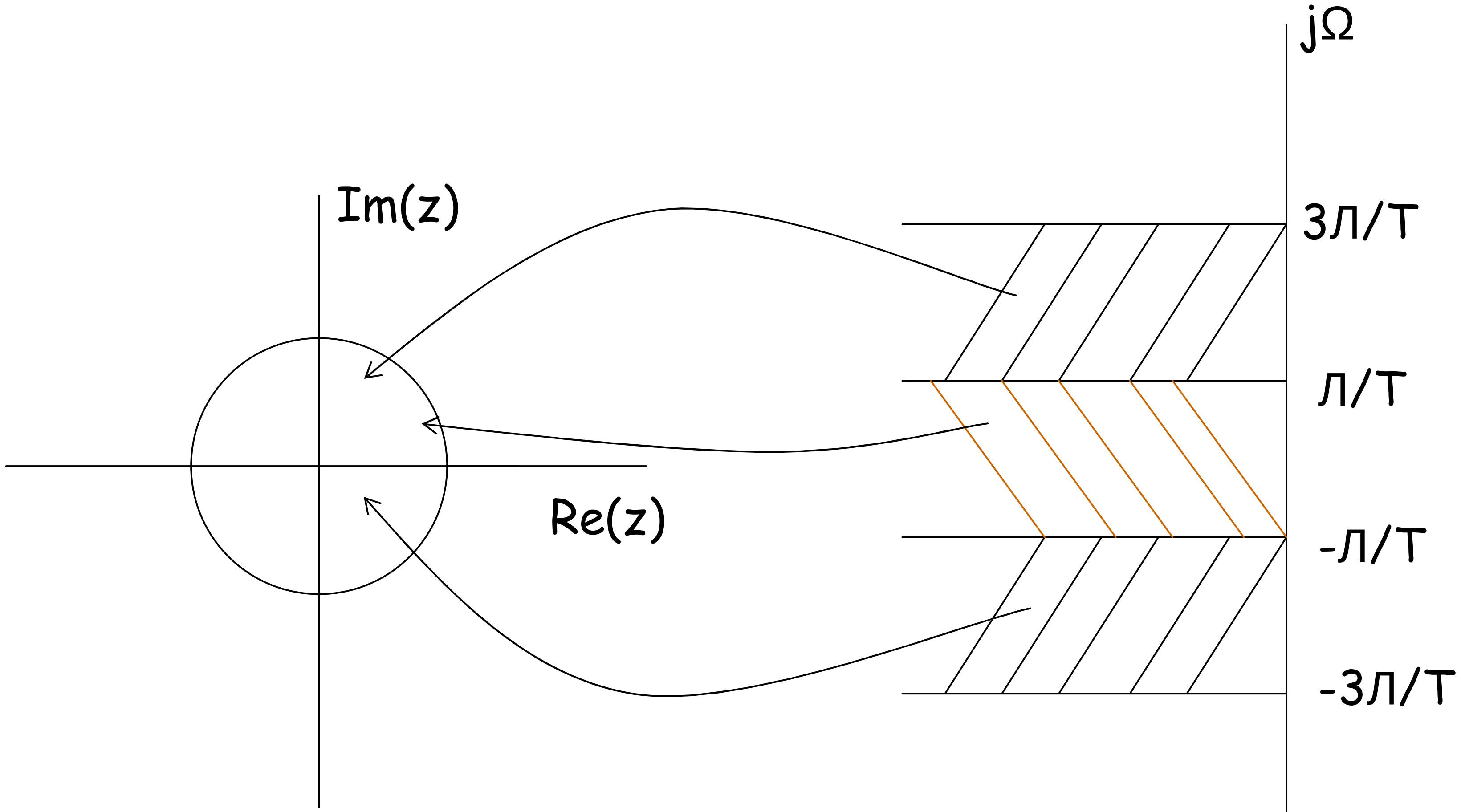
The interval $-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T}$ maps into $-\pi \le \omega \le \pi$

Also, the interval $\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T}$ maps into $-\pi \le \omega \le \pi$

In general, $\frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$ maps into $-\pi \le \omega \le \pi$

mapping of Ω to ω is many to one

→Reflects the effect of aliasing due to sampling


jΩ
Im(z)
3Л/T
Л/T
Re(z)
-Л/T
-3Л/T

Bilinear Transformation

Mapping from s-plane to z-plane is

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

$$\sigma + j\Omega = \frac{2}{T}\left(\frac{r^2-1}{1+2r\cos\omega + r^2}\right) + j\frac{2}{T}\left(\frac{2r\sin\omega}{1+2r\cos\omega + r^2}\right)$$

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{1 + 2r\cos\omega + r^2}\right)$$

$$\therefore if \quad r < 1, \quad \sigma < 0$$

$$r > 1, \quad \sigma > 0$$

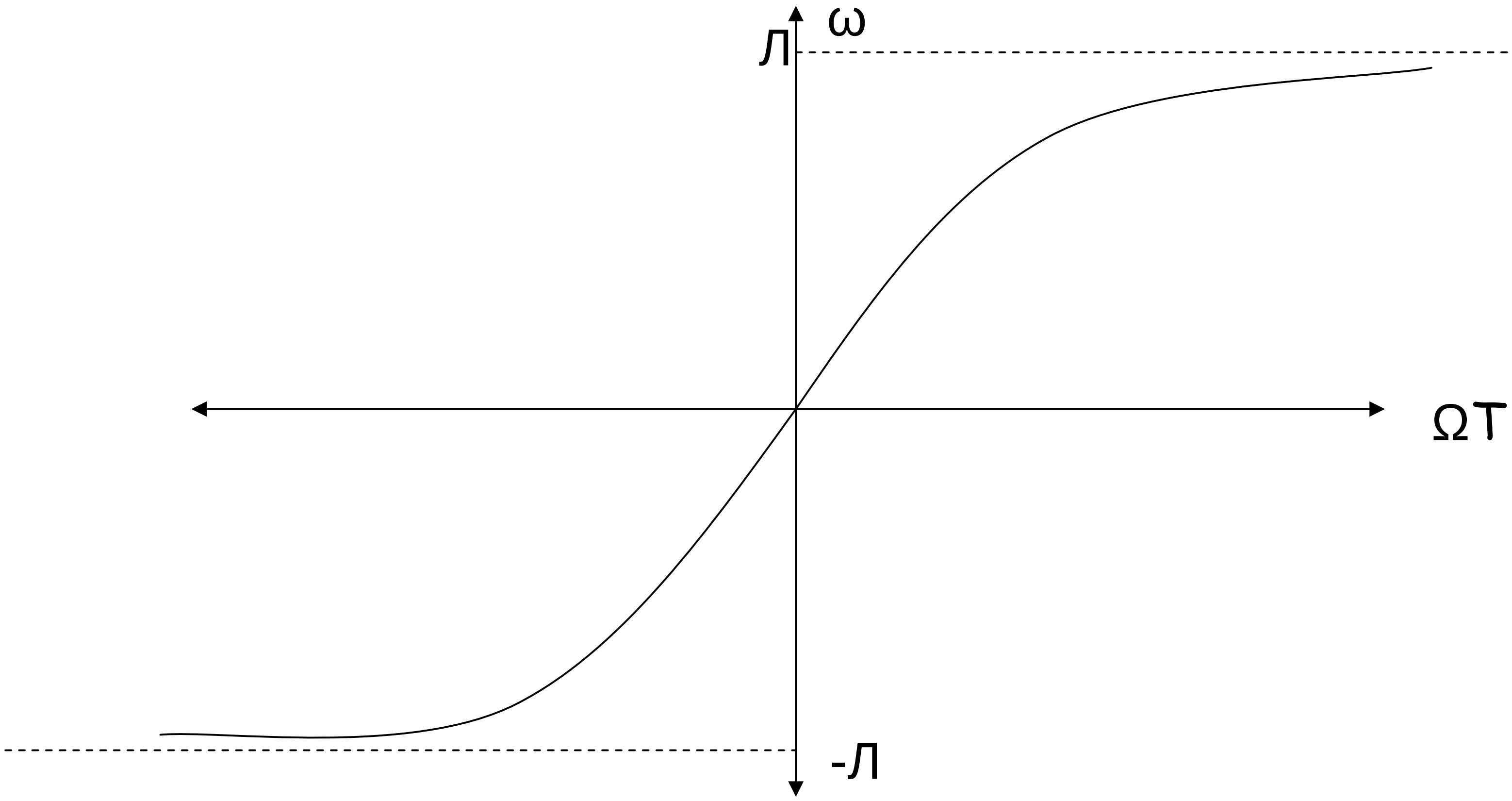
$$r = 1, \quad \sigma = 0$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+2r\cos\omega+r^2}\right)$$

$$At \quad r=1,\ \ \Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\therefore\ \ \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

$$\therefore\ \omega = 2\tan^{-1}\frac{\Omega T}{2}$$



- Entire range of Ω is mapped only once into the range $-\pi <= \omega <= \pi$ (one-to-one mapping)

- Mapping is highly non-linear
  - Low frequencies in analog domain are expanded in digital domain
  - High frequencies are compressed

- This is due to non-linearity of arctangent function

A nonlinear relationship between digital frequency and analog frequency leads to a distortion of frequency axis -
This phenomenon is called as **Frequency Warping**

Analog filter transfer function

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

N → Even

$$H_a(s) = \frac{B_0 \Omega_c^2}{s + c_0 \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

N → Odd

Butterworth Approximation

- Simplest approximation Based on squared magnitude function (SMF)

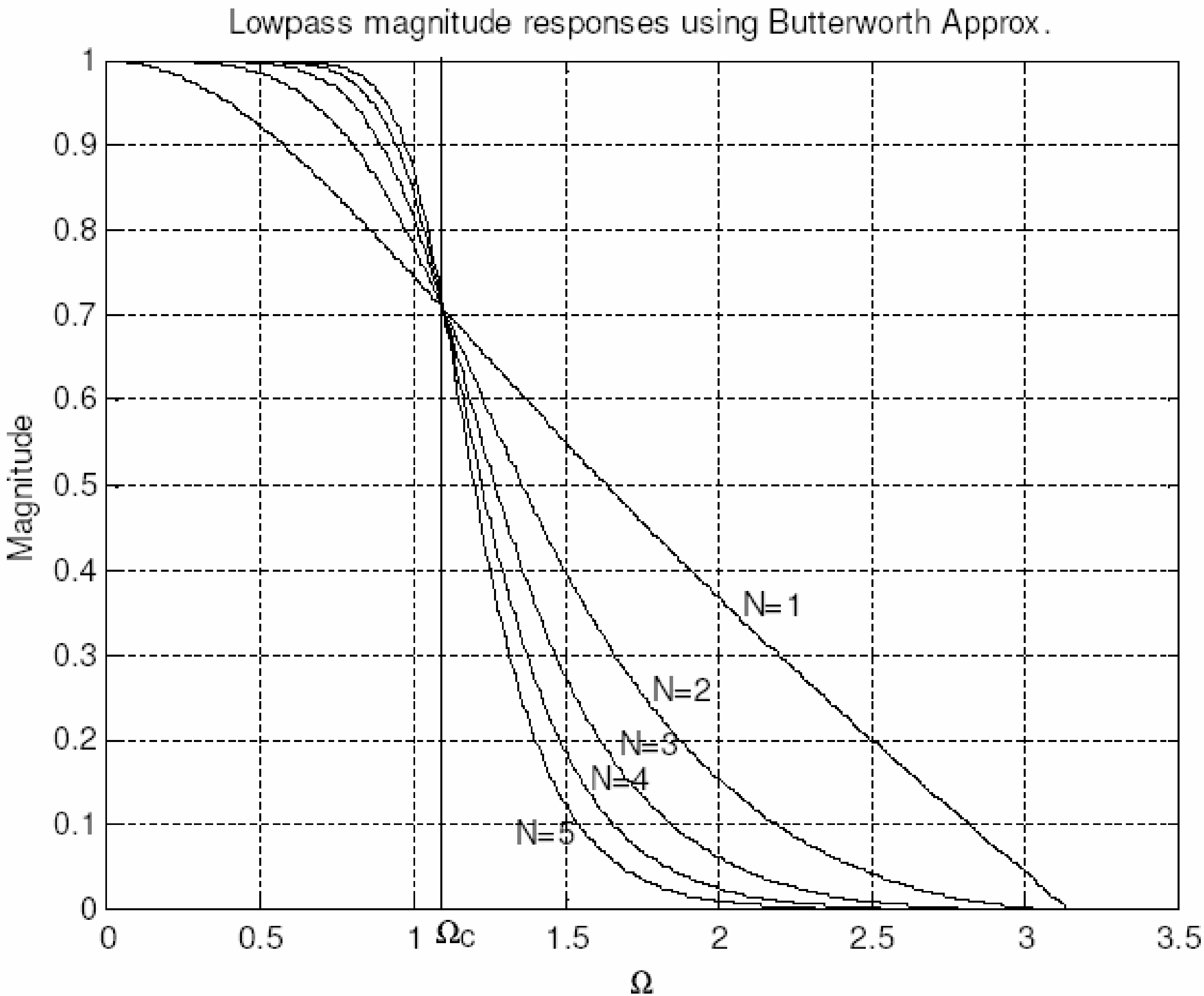
$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

or

$$|H_a(j\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

- Maximally Flat
- First 2N-1 derivatives of SMF are zero at $\Omega=0$
- Phase response become more non-linear as N increases

$$\Omega_c = \frac{\Omega_p}{\epsilon^{1/N}}$$


Lowpass magnitude responses using Butterworth Approx.
Magnitude
Ω
Ωc
N=1
N=2
N=3
N=4
N=5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5

Steps to Design IIR filter using Butterworth Approximation

Given filter specifications:
A_p → Min. gain in passband
ω_p → Passband edge frequency
A_{st} → Max. gain in stopband
ω_{st} → stopband edge frequency

1. Determine analog frequencies using prewarping
   Ω_p, Ω_{st}, or Ω_{st}/Ω_p

   Impulse invariant method : $\Omega = \omega/T$

   Bilinear Transformation : $\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$

2. Determine the order of the filter :

$$N \geq \frac{1}{2}\,\frac{\log\left[\dfrac{\left(\dfrac{1}{A_{st}^{2}}-1\right)}{\left(\dfrac{1}{A_{p}^{2}}-1\right)}\right]}{\log\left(\dfrac{\Omega_{st}}{\Omega_{p}}\right)}$$

2. If attenuation in dB is given :

$$N \geq \frac{1}{2} \frac{\log\left[\dfrac{10^{0.1A_{st}} - 1}{10^{0.1A_p} - 1}\right]}{\log\left(\dfrac{\Omega_{st}}{\Omega_p}\right)}$$

3. Determine 3 dB cut-off frequency Ω_c :

$$\Omega_c = \frac{\Omega_1}{\left(\dfrac{1}{A_p^2} - 1\right)^{\frac{1}{2N}}}$$

4. Determine N poles : Consider poles of left half of s-plane

$$p_k = \pm \Omega_c \, e^{\frac{j(N+2k+1)\pi}{2N}} \qquad k = 0,1,\ldots,N-1$$

5. Determine transfer function $H_a(s)$:

$$H(s) = \frac{\Omega_c{}^N}{(s-p_1)(s-p_1^*)(s-p_2)(s-p_2^*)\ldots(s-p_{n/2})(s-p_{n/2}^*)}$$

6. Determine H(z) using mapping function :

Impulse Invariance
or
Bilinear Transformation

Chebyshev Approximation (Type I)

- Optimum all pole approx.

  i.e. for a given order N and given passband and stopband constraints, no other all pole filter has a narrower bandwidth.

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \qquad N = 1,2,3\ldots$$

- All pole filter
- Equiripple behavior in passband
- Monotonic behavior stopband

$$C_N(x) = \text{Chebyshev Polynomial of first kind of degree N}$$

$$= \cos(N\cos^{-1}x) \qquad x < 1$$

$$= \cosh(N\cosh^{-1}x) \qquad x > 1$$

$$= 2xC_{N-1}(x) - C_{N-2}(x) \quad \text{Recursive relation}$$

$$\in^2 = \frac{1}{\left(1-\delta_p^2\right)} - 1$$

$$= \frac{1}{A_p^2} - 1$$

Steps to Design IIR filter using Chebyshev (Type I) Approximation

Given filter specifications:

$A_p \rightarrow$ Min. gain in passband

$\omega_p \rightarrow$ Passband edge frequency

$A_{st} \rightarrow$ Max. gain in stopband

$\omega_{st} \rightarrow$ stopband edge frequency

1. Determine analog frequencies using prewarping
   Ω_p, Ω_{st}, or Ω_{st}/Ω_p

   Impulse invariant method : $\Omega = \omega/T$

   Bilinear Transformation : $$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

2. Determine the order of the filter :

$$N \geq \frac{\cosh^{-1}\left[\left(\frac{1}{\in}\right)\left(\frac{1}{A_{st}^2}-1\right)^{1/2}\right]}{\cosh^{-1}\left(\frac{\Omega_{st}}{\Omega_p}\right)}$$

If attenuation in dB is given :

$$N \geq \frac{1}{2}\,\frac{\cosh^{-1}\left[\frac{10^{0.1A_{st}}-1}{10^{0.1A_p}-1}\right]}{\cosh^{-1}\left(\frac{\Omega_{st}}{\Omega_p}\right)}$$

3. Determine 3 dB cut-off frequency Ω_c :

$$\Omega_c = \Omega_p \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\in}\right)\right]$$

4. Determine transfer function $H_a(s)$:

$$H_a(s) = \frac{B_0 \Omega_c^2}{s + c_0 \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

N → Odd

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

N → Even

Determine B_k :

$$\frac{1}{\left(1+\epsilon^2\right)^{0.5}} = \prod_{k=1}^{N/2} \frac{B_k}{c_k}$$

N → even

$$1 = \prod_{k=1}^{(N-1)/2} \frac{B_k}{c_k}$$

N → Odd

$$c_0 = Y_N$$

$$b_k = 2Y_N \sin\left(\frac{2k-1}{2N}\right)\pi$$

$$c_k = Y_N^2 + \cos^2\left(\frac{2k-1}{2N}\right)\pi$$

$$Y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{1/N} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{-1/N}\right\}$$

5. Determine H(z) using mapping function :

Impulse Invariance

or

Bilinear Transformation

Analog Frequency Transformations

Low pass to Low pass

$$s = \frac{\Omega_p s}{\Omega'_p}$$

$$s = \frac{s}{\Omega'_p}$$

Low pass to High pass

$$s = \frac{\Omega_p \Omega'_p}{s}$$

$$s = \frac{\Omega'_p}{s}$$

Low pass to Band pass

$$s = \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$s = \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

Low pass to Band stop

$$s = \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

$$s = \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

Prototype <u>low pass</u> from <u>high</u> pass specs.

Given :
Pass band edge frequency: Ω_p
Stop band edge frequency: Ω_s

Prototype low pass specs.:

Pass band edge frequency: $\Omega^p{}_p=1$

Stop band edge frequency: $\Omega^p{}_s=\Omega_p/\Omega_s$

Prototype low pass from band pass specs.

Given :

Lower Pass band edge frequency: Ω_{pl}

Lower Stop band edge frequency: Ω_{sl}

Higher Pass band edge frequency: Ω_{h}

Higher Stop band edge frequency: Ω_{sh}

$$\Omega_s^p = \frac{\Omega_{sh}^2 - \Omega_0^2}{W\Omega_{sh}}$$

Center Frequency : Ω_0

Prototype low pass specs.:

$$\Omega_0^2 = \Omega_{pl}\Omega_{ph}$$

Pass band edge frequency: $\Omega^p{}_p=1$

Stop band edge frequency:

$$W = \Omega_{ph} - \Omega_{pl}$$

Prototype low pass from band stop specs.

Given :
Lower Pass band edge frequency: Ω_{pl}
Lower Stop band edge frequency: Ω_{sl}

Higher Pass band edge frequency: Ω_{ph}
Higher Stop band edge frequency: Ω_{sh}

$$\Omega_s^p = \frac{W\Omega_{sl}}{\Omega_0^2 - \Omega_{sl}^2}$$

Center Frequency : Ω_0

Prototype low pass specs.:

$$\Omega_0^2 = \Omega_{ph}\Omega_{pl}$$

Pass band edge frequency: $\Omega^p_p=1$
Stop band edge frequency:

$$W = \Omega_{ph} - \Omega_{pl}$$